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ON PRO-ISOMORPHIC ZETA FUNCTIONS OF D^*-GROUPS OF EVEN HIRSCH LENGTH*

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ABSTRACT

The pro-isomorphic zeta function of a finitely generated nilpotent group is a Dirichlet generating series that enumerates all finite-index subgroups whose profinite completion is isomorphic to that of the ambient group. We study the pro-isomorphic zeta functions of $\mathbb{Q}$-indecomposable D^*-groups of even Hirsch length. These groups are building blocks of finitely generated class-two nilpotent groups with rank-two centre, up to commensurability. Due to a classification by Grunewald and Segal, they are parameterised by primary polynomials whose companion matrices define commutator relations for an explicit presentation. For Grunewald–Segal representatives of even Hirsch length of type $f(t) = t^m$, we give a complete description of the algebraic automorphism groups of associated Lie lattices. Utilising the automorphism groups, we determine the local pro-isomorphic zeta functions of groups associated to t^2 and t^3. In both cases, the local zeta functions are uniform in the prime p and satisfy functional equations. The functional equations for these groups, not predicted by the currently available theory, prompt us to formulate a conjecture which prescribes, in particular, information about the symmetry factor appearing in local functional equations for pro-isomorphic zeta functions of nilpotent groups. Our description of the local zeta functions also yields information about the analytic properties of the corresponding global pro-isomorphic zeta functions. Some of our results for the D^*-groups associated to t^2 and t^3 generalise to two infinite families of class-two nilpotent groups that result naturally from the initial groups via 'base extensions'.

1. Introduction

1.1. Setting the scene. Zeta functions of groups and rings were introduced by Grunewald, Segal and Smith [19] as an effective means for studying subgroup growth. Since their inception in the late 1980s, much progress has been made regarding their analytic and arithmetic properties; see for instance [13, 37]. In this paper we focus on pro-isomorphic zeta functions. Let Γ be a finitely generated nilpotent group and let $a_n^\wedge(\Gamma)$ denote the number of subgroups $\Delta \leq \Gamma$ satisfying $|\Gamma : \Delta| = n$ and $\widehat{\Delta} \cong \widehat{\Gamma}$, where $\widehat{H}$ denotes the profinite completion of a group H. The **pro-isomorphic zeta function** of Γ is the Dirichlet generating series

$$\zeta_\Gamma^\wedge(s) = \sum_{n=1}^{\infty} a_n^\wedge(\Gamma)\, n^{-s} \quad (s \in \mathbb{C}).$$

As with subgroup and normal subgroup zeta functions, an immediate consequence of nilpotency is that the pro-isomorphic zeta function has an Euler product decomposition over all rational primes:

$$\zeta^{\wedge}_{\Gamma}(s) = \prod_{p} \zeta^{\wedge}_{\Gamma,p}(s), \quad \text{where } \zeta^{\wedge}_{\Gamma,p}(s) = \sum_{k=0}^{\infty} a^{\wedge}_{p^k}(\Gamma)\, p^{-ks} \tag{1.1}$$

is called the **local zeta function** at a prime p and is known to be a rational function in p^{-s} over $\mathbb{Q}$; see [19].

A special feature of pro-isomorphic zeta functions, in contrast to other related zeta functions of groups, is that the local zeta functions can be expressed rather naturally as p-adic integrals over algebraic groups taking the form

$$Z(\mathbf{G}, p)(s) = \int_{G_p^+} |\det(g)|_p^s \, \mathrm{d}\mu_p(g). \tag{1.2}$$

Here $\mathbf{G} \leq \mathsf{GL}_d$ is an affine $\mathbb{Z}$-group scheme (the algebraic automorphism group $\mathbf{Aut}(L)$ of an associated nilpotent Lie lattice L), μ_p denotes a suitably normalised Haar measure on the locally compact p-adic group $G_p = \mathbf{G}(\mathbb{Q}_p)$, and $G_p^+ = G_p \cap \mathsf{M}_d(\mathbb{Z}_p)$ is a compact open subset of G_p; the precise details are described in Section 3.

Integrals such as (1.2) have a long history and were studied for various classical groups by Hey, Weil, Tamagawa, Igusa and others [20, 39, 33, 23]; for a more detailed account see [14]. Grunewald, Segal and Smith [19] discovered the relevance of such integrals for the study of pro-isomorphic zeta functions. Subsequently, du Sautoy and Lubotzky [14] advanced the general theory of integrals of the form (1.2) by considering non-reductive groups $\mathbf{G}$; an essential aspect of their work was to carry out a reduction of the integral, subject to certain technical assumptions, to an integral over a reductive subgroup.

It is remarkable that in many cases (e.g., when the algebraic group $\mathbf{G}$ is irreducibly reductive and split over $\mathbb{Q}$) the zeta functions $Z_p(s) = Z(\mathbf{G}, p)(s)$ are given by a single rational function in p, p^{-s} and display a symmetry upon inversion of the prime, for almost all primes p:

$$Z_p(s)_{p \to p^{-1}} = (-1)^j p^{a-bs} Z_p(s) \quad \text{for suitable } a, b, j \in \mathbb{N}_0.$$

Constructions using base extensions lead to slightly more general situations, where the zeta functions are finitely uniform and a corresponding finite variation $a = a(p), b = b(p)$ with p is observed; compare with [19, Thm. 4], [14, §3] and [8]. In these contexts the functional equation is a manifestation of the

compatibility of the integral with the p-adic Bruhat decomposition and symmetries related to the affine Weyl group of the reductive group $\mathbf{G}$; see [23, 14]. Such a phenomenon should be compared with the symmetries conjectured by Igusa and proved by Denef and Meuser [10] for integrals over $\mathbb{Z}_p^d$ of integral homogeneous polynomials, based on the principalisation of ideals and the Weil conjectures. More general results in this direction, with group-theoretic applications, were discovered and proved by Voll [36]. Since then functional equations of the kind discussed have been recognised as a widespread, but not universal feature of zeta functions associated to groups, rings and modules; for instance, see [1, 32, 29, 38, 25, 15, 5].

1.2. Main results and a conjecture. The motivations for the present paper are two-fold. Firstly, we wish to explore pro-isomorphic zeta functions of nilpotent groups in situations where a crucial standard assumption, originally introduced in [14] and until now widely used to study integrals of the form (1.2), does not hold. For this purpose, we consider finitely generated torsion-free class-two nilpotent groups with rank-two centres; we refer to such groups as **D^*-groups**. An explicit example from this family is studied in depth in this paper, pertaining to the D^*-group Γ_{t^3} of Hirsch length 8, associated to the primary polynomial t^3; see Theorem 1.3 below and the following discussion. Our analysis relies in the first place on pinning down the automorphism group of Γ_{t^3}. More generally, we extend the computation, initiated in [6], of the automorphism groups of Grunewald–Segal representatives of $\mathbb{Q}$-indecomposable D^*-groups, up to commensurability; see Theorem 1.10. In addition to its inherent interest, our description of the automorphism groups provides a first essential step toward studying the pro-isomorphic zeta functions of more complicated D^*-groups; we extend our description of the relevant automorphism groups further in [7]. Indeed, after our original work was finished, Moadim Lesimcha and Schein [28] went ahead and studied other families of D^*-groups; they produced a combinatorial description of local pro-isomorphic zeta functions and derived local functional equations for the families that they considered. Secondly, we wish to establish a conjectural framework for the shape that local functional equations take in the context of pro-isomorphic zeta functions of nilpotent groups, when they occur; see Conjecture 1.8.

We now provide more details. In [18, §6], Grunewald and Segal considered $\mathcal{D}^*$-groups, that is, torsion-free radicable class-two nilpotent groups of finite

rank with rank-two centres. They classified the indecomposable constituents of such groups, by giving a parametrisation in terms of the rank and—in even rank—an extra datum, namely the projective equivalence class of an associated binary form over $\mathbb{Q}$. Each $\mathcal{D}^*$-group is the radicable hull of a D^*-group, determined up to commensurability. We refer to such 'integral representatives' of indecomposable $\mathcal{D}^*$-groups as **$\mathbb{Q}$-indecomposable D^*-groups**.

In [18, Thm. 6.3], Grunewald and Segal effectively gave explicit presentations for certain $\mathbb{Q}$-indecomposable D^*-groups, which cover all such groups up to commensurability. For convenience, we refer to these special groups as Grunewald–Segal representatives. In passing, we remark that the local normal subgroup zeta functions of such Grunewald–Segal representatives were computed in [35, §3.2]. The automorphism groups of Grunewald–Segal representatives for $\mathbb{Q}$-indecomposable D^*-groups of odd Hirsch length were determined in [6]. In the current paper we consider Grunewald–Segal representatives for $\mathbb{Q}$-indecomposable D^*-groups of even Hirsch length; these are defined explicitly in Section 2. We are particularly interested in a subfamily of D^*-groups Γ_{t^m}, $m \in \mathbb{N}$, given by the presentations

$$\begin{aligned}\Gamma_{t^m} = \langle x_1, \dots, x_m, y_1, \dots, y_m, z_1, z_2 \mid\ & \\ & [x_i, y_i] = z_1 \text{ for } 1 \le i \le m, \\ & [x_j, y_{j+1}] = z_2 \text{ for } 1 \le j < m, \\ & [x_i, y_j] = 1 \text{ for } 1 \le i, j \le m \text{ with } j - i \notin \{0, 1\}, \\ & [x_i, x_j] = [y_i, y_j] = [x_i, z_1] = [x_i, z_2] = [y_i, z_1] = [y_i, z_2] = 1 \\ & \qquad \text{for } 1 \le i, j \le m \rangle.\end{aligned} \tag{1.3}$$

Observe that Γ_{t^m} has Hirsch length $2m+2$ and rank-two centre $\mathrm{Z}(\Gamma_{t^m}) = \langle z_1, z_2 \rangle$. For $m=1$, the presentation yields the decomposable D^*-group

$$\Gamma_t \cong C_\infty \times \mathrm{Heis}(\mathbb{Z}),$$

the direct product of an infinite cyclic group and the discrete Heisenberg group. Its pro-isomorphic zeta function is relatively easy to compute:

$$\zeta^{\wedge}_{\Gamma_t}(s) = \zeta(s-2)\zeta(2s-3)\zeta(2s-4)$$

is a product of shifted Riemann zeta functions; this case was already treated in [3, §3.3.4] and we confirm the result in Example 3.8.

For $m \geq 2$, the groups Γ_{t^m} constitute one basic family of Grunewald–Segal representatives for $\mathbb{Q}$-indecomposable D^*-groups. In Theorem 1.10 below we provide, for all $m \in \mathbb{N}$, a complete description of the algebraic automorphism groups of associated Lie lattices. Based on this result, we explicitly determine for $m \in \{2, 3\}$ the corresponding pro-isomorphic zeta functions, including all local zeta functions with no exceptions.

THEOREM 1.1: *For all primes p, the D^*-group $\Gamma = \Gamma_{t^2}$ satisfies*

$$\zeta^{\wedge}_{\Gamma,p}(s) = W_{t^2}(p, p^{-s}),$$

where

$$W_{t^2}(X, Y) = \frac{1 + X^{10}Y^4}{(1 - X^8Y^3)(1 - X^{11}Y^4)(1 - X^{12}Y^5)}.$$

Thus $\zeta^{\wedge}_{\Gamma,p}(s)$ has abscissa of convergence $11/4$ and satisfies the functional equation

$$\zeta^{\wedge}_{\Gamma,p}(s)|_{p \to p^{-1}} = (-1)p^{21-8s}\, \zeta^{\wedge}_{\Gamma,p}(s).$$

COROLLARY 1.2: *The pro-isomorphic zeta function of the D^*-group $\Gamma = \Gamma_{t^2}$ is*

$$\zeta^{\wedge}_{\Gamma}(s) = \frac{\zeta(3s-8)\zeta(4s-11)\zeta(5s-12)\zeta(4s-10)}{\zeta(8s-20)},$$

where $\zeta(s)$ denotes the Riemann zeta function; in particular, it admits meromorphic continuation to the entire complex plane and has abscissa of convergence 3, with a double pole at $s = 3$.

Furthermore, the asymptotic growth of pro-isomorphic subgroups in Γ is given by

$$\sum_{n=1}^{N} a^{\wedge}_n(\Gamma) \sim c_{t^2} N^3 \log N \quad \text{as } N \to \infty, \tag{1.4}$$

where

$$c_{t^2} = \frac{5\,\zeta(3)}{12\pi^2} \approx 0.050747.$$

Theorem 1.1 and its proof resemble similar results for other nilpotent groups, for instance the D^*-groups studied in [6]. In contrast, the next theorem and its proof open up several promising new directions for further exploration.

THEOREM 1.3: *For all primes p, the D^*-group $\Gamma = \Gamma_{t^3}$ satisfies*

$$\zeta^{\wedge}_{\Gamma,p}(s) = W_{t^3}(p, p^{-s}),$$

where

$$W_{t^3}(X,Y)=(1-X^{29}Y^{10})$$
$$\times\frac{(1+X^{14}Y^5-X^{15}Y^5+X^{30}Y^{10}-X^{59}Y^{21}+X^{74}Y^{26}-X^{75}Y^{26}-X^{89}Y^{31})}{(1-X^{15}Y^5)^2(1-X^{29}Y^9)(1-X^{30}Y^{11})(1-X^{61}Y^{21})}.$$

Thus $\zeta^{\wedge}_{\Gamma,p}(s)$ has abscissa of convergence $29/9$ and satisfies the functional equation

$$\zeta^{\wedge}_{\Gamma,p}(s)|_{p\to p^{-1}} = (-1)p^{32-10s}\,\zeta^{\wedge}_{\Gamma,p}(s).$$

COROLLARY 1.4: *The pro-isomorphic zeta function of the D^*-group $\Gamma=\Gamma_{t^3}$ has abscissa of convergence $10/3$ and admits meromorphic continuation to $\{s\in\mathbb{C}\mid \mathrm{Re}(s)>3\}$ via*

$$\zeta^{\wedge}_{\Gamma}(s)=\frac{\zeta(5s-15)\zeta(9s-29)\zeta(10s-30)\zeta(11s-30)\zeta(15s-45)\zeta(21s-61)}{\zeta(10s-29)\zeta(30s-90)}\widetilde{\psi}(s),$$

where $\zeta(s)$ denotes the Riemann zeta function and

$$\widetilde{\psi}(s)=\prod_p \frac{\widetilde{W}(p,p^{-s})}{1-p^{15-5s}+p^{30-10s}}$$

for

$$\widetilde{W}(X,Y)=1+X^{14}Y^5-X^{15}Y^5+X^{30}Y^{10}-X^{59}Y^{21}+X^{74}Y^{26}-X^{75}Y^{26}-X^{89}Y^{31};$$

moreover, the line $\{s\in\mathbb{C}\mid \mathrm{Re}(s)=3\}$ is a natural boundary. In particular, the zeta function $\zeta^{\wedge}_{\Gamma}(s)$ has a simple pole at $s=10/3$.

Remark 1.5: Similar to Corollary 1.2, the asymptotic growth of pro-isomorphic subgroups in $\Gamma=\Gamma_{t^3}$ can be described by means of a suitable Tauberian theorem:

$$\sum_{n=1}^{N} a_n^{\wedge}(\Gamma)\sim c_{t^3}N^{10/3}\quad\text{as } N\to\infty,$$

where

$$c_{t^3}=\frac{\zeta(5/3)\,\zeta(10/3)\,\zeta(20/3)\,\zeta(5)\,\zeta(9)\,\widetilde{\psi}(10/3)}{30\,\zeta(13/3)\,\zeta(10)}\in\mathbb{R}_{>0}$$

is somewhat unwieldy.

Following a suggestion of the referee, in Section 7 we extend our results for the $\mathbb{Q}$-indecomposable D^*-groups Γ_{t^2} and Γ_{t^3} to two infinite families, $\widetilde{\Gamma}_{t^2,k}$ and $\widetilde{\Gamma}_{t^3,k}$ of class-two nilpotent groups, where k runs through all number fields. These families of groups result naturally from the initial groups via 'base extensions' of corresponding Lie lattices, and pro-isomorphic zeta functions of groups constructed in this way were systematically investigated in [8]. For completeness we also discuss the family $\widetilde{\Gamma}_{t,k}$ associated to the decomposable D^*-group Γ_t. We state here the generalisation of Corollary 1.2; further details about the set-up and generalisations of some of our other results can be found in Section 7.

THEOREM 1.6: *Let k be a number field of absolute degree $d = [k : \mathbb{Q}]$, with ring of integers $\mathfrak{o}$. Let $\widetilde{\Gamma} = \widetilde{\Gamma}_{t^2,k}$ be the class-two nilpotent group of Hirsch length $6d$ and with rank-$2d$ centre, corresponding to the class-two nilpotent $\mathbb{Z}$-Lie lattice $\widetilde{L} = \widetilde{L}_{t^2,k}$ which results from the Lie lattice $L = L_{t^2}$ associated to the group Γ_{t^2} by extension of scalars from $\mathbb{Z}$ to $\mathfrak{o}$ and subsequent restriction of scalars back to $\mathbb{Z}$.*

Then the pro-isomorphic zeta function of the group $\widetilde{\Gamma}$ is

$$\zeta^{\wedge}_{\widetilde{\Gamma}}(s) = \frac{\zeta_k(3s-(4d+4))\zeta_k(4s-(8d+3))\zeta_k(5s-12d)\zeta_k(4s-(8d+2))}{\zeta_k(8s-(16d+4))}, \tag{1.5}$$

where $\zeta_k(s)$ denotes the Dedekind zeta function of k; in particular, it admits meromorphic continuation to the entire complex plane.

Remark 1.7: For $k = \mathbb{Q}$, i.e., $d = 1$, we recover Corollary 1.2. For quadratic fields k, i.e., $d = 2$, the abscissa of convergence is 5, with a double pole at $s = 5$. For number fields k of absolute degree $d \geq 3$, the abscissa of convergence is $(12d+1)/5$, with a simple pole at $s = (12d+1)/5$. Similar to Corollary 1.2, the asymptotic growth of pro-isomorphic subgroups in $\widetilde{\Gamma}$ can be described by means of a suitable Tauberian theorem. Via the Euler product, the formula (1.5) incorporates a description of the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)$ for all primes p and thus also yields a generalisation of Theorem 1.1. Indeed, for $d \geq 2$ the zeta function $\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)$ has abscissa of convergence $12d/5$ and, if p is unramified in k, it satisfies the functional equation

$$\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)|_{p\to p^{-1}} = \pm p^{16d^2+5d-8ds}\, \zeta^{\wedge}_{\widetilde{\Gamma},p}(s).$$

Theorem 1.3 and its proof extend the scope of functional equations and the complexity of the integrals arising in the context of pro-isomorphic zeta functions of class-two nilpotent groups. As alluded to above, and demonstrated in Remark 5.6 below, it is the first explicitly computed pro-isomorphic zeta function for which a certain lifting condition [14, Assumption 2.3] does not hold. Furthermore, it involves a technically challenging computation of an integral with non-multiplicative integrand which requires careful analysis by certain number-theoretic and combinatorial techniques. In particular, one needs to count solutions to congruence equations of the form $p^\alpha x^2 + p^\beta yz \equiv 0 \mod p^n$; see Section 5. This reveals a new phenomenon in the setting of pro-isomorphic zeta functions, namely the prominent role played by counting points on reductions of varieties; previously this feature was encountered only for other types of zeta functions of nilpotent groups, such as subgroup and normal subgroup zeta functions; compare with [12, 11, 36]. Our analysis of the structure of the automorphism groups of $\mathbb{Q}$-indecomposable D^*-groups of even Hirsch length given in Section 2 suggests that this is only the tip of the iceberg, and should be contrasted with the linearity assumption in [14, §5].

The available theory on integrals of the form (1.2), which occupy a central role in our computation, could not be used to predict a priori the resulting form of the local pro-isomorphic zeta function in any sense. It is thus somewhat of a surprise that the zeta functions in Theorem 1.3 satisfy local functional equations. In contrast to the situation for $\mathbb{Q}$-indecomposable D^*-groups of odd Hirsch length [6], the values of the abscissae of convergence—for the pro-isomorphic zeta functions of $\mathbb{Q}$-indecomposable D^*-groups of even Hirsch length—remain elusive. More work is required, even to produce a promising conjecture for the family of groups Γ_{t^m}, $m \in \mathbb{N}_{\geq 2}$.

In order to compare the local functional equations in Theorems 1.1, 1.3 and their generalisations with data for other groups, we briefly recall further concepts. To a finitely generated torsion-free class-c nilpotent group Γ of Hirsch length d one associates, via Lie theory, a class-c nilpotent $\mathbb{Z}$-Lie lattice L of $\mathbb{Z}$-rank d, whose local zeta functions

$$\zeta^{\wedge}_{L,p}(s) = \zeta^{\mathrm{iso}}_{L_p}(s) = \sum_{k=0}^{\infty} a^{\mathrm{iso}}_{p^k}(L_p) p^{-ks}$$

satisfy

$$\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s)$$

for almost all primes p; here $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$ denotes the p-adic completion of L, and $a_{p^k}^{\mathrm{iso}}(L_p)$ is the number of Lie sublattices of L_p of index p^k which are isomorphic to L_p. It was shown in [19] that each local zeta function $\zeta_{L,p}^{\wedge}(s)$ is a rational function in p^{-s} over $\mathbb{Q}$, i.e., $\zeta_{L,p}^{\wedge}(s) = W_p(p^{-s})$ for suitable $W_p = R_p/Q_p$ with $R_p, Q_p \in \mathbb{Q}[Y]$. We then define the **degree** of a local pro-isomorphic zeta function, denoted by $\deg_{p^{-s}} \zeta_{L,p}^{\wedge}(s)$, to be the degree of the rational function W_p, viz.

$$\deg W_p = \deg_Y R_p - \deg_Y Q_p.$$

The family of local zeta functions $\zeta_{L,p}^{\wedge}(s)$ is said to be **finitely uniform** if there exist finitely many rational functions $\mathcal{W}_1, \ldots, \mathcal{W}_r \in \mathbb{Q}(X, Y)$ in two variables such that, for each prime p, there is an index $i = i(p)$ for which

$$\zeta_{L,p}^{\wedge}(s) = W_p(p^{-s}) = \mathcal{W}_i(p, p^{-s}).$$

Another ingredient relates to the nilpotent $\mathbb{Z}$-Lie lattice L itself: recall that L is $\mathbb{N}$**-graded** if it is equipped with an additive decomposition $L = \bigoplus_{i \in \mathbb{N}} L_{(i)}$ such that $[L_{(i)}, L_{(j)}] \subseteq L_{(i+j)}$ for all $i, j \in \mathbb{N}$; for short, we refer to the latter as a grading on L. Since L has finite rank as a $\mathbb{Z}$-module, there exists, for a given grading, a minimal $l \in \mathbb{N}_0$ such that $L_{(j)} = 0$ for $j > l$; the grading then gives rise to a descending filtration $L = \overline{L}_{(1)} \supseteq \overline{L}_{(2)} \supseteq \cdots \supseteq \overline{L}_{(l)} \supseteq \{0\}$ of L by Lie sublattices $\overline{L}_{(i)} = \sum_{j=i}^{l} L_{(j)} \supseteq \gamma_i(L)$. We call a grading **natural** if its associated filtration is precisely the lower central series, i.e., if $\overline{L}_{(i)} = \gamma_i(L)$ for $1 \leq i \leq l$ and $l = c$ is the nilpotency class of L. To a grading on L as above we attach a **weight** given by

$$\sum_{i=1}^{l} i \ \operatorname{rk}_{\mathbb{Z}} L_{(i)} = \sum_{i=1}^{l} \operatorname{rk}_{\mathbb{Z}} \overline{L}_{(i)},$$

and we call a grading **minimal** if its weight is minimal amongst all weights of gradings on L. In passing, we mention that not all nilpotent Lie lattices admit a grading. For instance, Dyer [16] constructed a 9-dimensional class-6 nilpotent Lie algebra over $\mathbb{Q}$ whose algebraic automorphism group is unipotent. This implies that the Lie algebra does not possess any grading, since every non-zero graded Lie algebra admits non-trivial semisimple automorphisms; clearly, no Lie lattice in such a Lie algebra can possess a grading.

CONJECTURE 1.8: *Let L be a nilpotent $\mathbb{Z}$-Lie lattice that admits at least one grading. Then, for almost all primes p, the degree of the local pro-isomorphic zeta function of L at p is equal to the weight of a minimal grading of L.*

In particular, if the family of local pro-isomorphic zeta functions $\zeta^{\wedge}_{L,p}(s)$ is finitely uniform and the local zeta functions satisfy, for almost all primes p, functional equations of the form

$$\zeta^{\wedge}_{L,p}(s)|_{p\to p^{-1}} = (-1)^j p^{a-bs}\zeta^{\wedge}_{L,p}(s) \quad \text{for suitable } a=a(p), b=b(p),\ j=j(p)\in\mathbb{N}_0,$$

then the integer b in the 'symmetry factor' is the same for almost all p and is given by the weight of a minimal grading of L.

Remark 1.9: Note that natural gradings, when they exist, are minimal. It follows that, if a class-c nilpotent Lie lattice L is naturally graded, then—in accordance with the conjecture—we expect that

$$\deg_{p^{-s}} \zeta^{\wedge}_{L,p}(s) = \sum_{j=1}^{c} \operatorname{rk}_{\mathbb{Z}} \gamma_j(L)$$

for almost all primes p. It is curious that this expression already has an interpretation in asymptotic group theory: it provides the degree of polynomial word growth of finitely generated nilpotent groups Γ giving rise to L via Lie theory; see [2]. In particular, every class-two nilpotent Lie lattice L is naturally graded and thus we expect that the degrees satisfy

$$\deg_{p^{-s}} \zeta^{\wedge}_{L,p}(s) = \operatorname{rk}_{\mathbb{Z}} L + \operatorname{rk}_{\mathbb{Z}}[L,L]$$

for almost all primes p.

In spirit, Conjecture 1.8 is similar to part of a conjecture of Voll on submodule zeta functions [38, Conj. 1.11], but the conjectures involve different types of filtrations (which can be seen already for the group Γ_t, arising from (1.3) for $m = 1$) and as yet there is no direct link between the two. We have tested Conjecture 1.8 comprehensively for all nilpotent $\mathbb{Z}$-Lie lattices L for which the local pro-isomorphic zeta functions are known; this list includes many naturally graded Lie lattices as well as some Lie lattices not possessing a natural grading; we refer to [19, 3, 6, 8, 28] for descriptions of relevant nilpotent $\mathbb{Z}$-Lie lattices and their pro-isomorphic zeta functions. The current paper provides two new infinite families of groups confirming the conjecture: the integers b in the symmetry factors of the local zeta functions described in Remarks 1.7 and 7.6 indeed match the sum of the ranks of terms of the lower central series: for the 'base extensions' defined in Theorems 1.6 and 7.5 one has $8d = 6d + 2d$ and $10d = 8d + 2d$ for all primes unramified in the extension.

Our conjecture also holds true for a $\mathbb{Z}$-Lie lattice L, constructed by Berman and Klopsch in [5], with the property that its local pro-isomorphic zeta functions $\zeta^{\wedge}_{L,p}(s)$ do not satisfy functional equations for $p > 3$. The relevant Lie lattice L is not naturally graded, but admits a minimal grading of weight 102; and, indeed, the local zeta functions are uniform in p, for $p > 3$, of degree 102. This example can also be generalised by means of base extensions; see [8].

It is well known and easy to see that there is a link between the existence of gradings of a $\mathbb{Z}$-Lie lattice L and the occurrence of diagonalisable elements in the algebraic automorphism group $\mathbf{Aut}(L)$ of L. Conjecture 1.8 suggests that there is a somewhat more delicate connection (yet to be discovered) between minimal gradings of a nilpotent $\mathbb{Z}$-Lie lattice L and the degrees of its local pro-isomorphic zeta functions, which stand in close relation to $\mathbf{Aut}(L)$ as indicated in (1.2).

In order to carry out the computations leading to Theorems 1.1 and 1.3 and their generalisations we require a structural description of the relevant automorphism groups. In fact, we determine the algebraic automorphism groups for the Lie lattices associated to Grunewald–Segal representatives of $\mathbb{Q}$-indecomposable D^*-groups of even Hirsch length associated to the primary polynomials $\Delta(t)=t^m$, for all $m \in \mathbb{N}$; as in the case of odd Hirsch length [6], this structure theorem for the algebraic automorphism groups is of independent interest. The presentation (1.3) for the group Γ_{t^m} readily translates into a description (2.2) of the corresponding Lie lattice; compare with Section 3.1.

THEOREM 1.10: *For $m \in \mathbb{N}$, let $\mathbf{G} \leq \mathsf{GL}_{2m+2}$ be the algebraic automorphism group of the $\mathbb{Z}$-Lie lattice (scheme) L associated, via (2.2) below, to the primary polynomial $\Delta(t) = t^m$. Let $\mathbf{G}_0 \trianglelefteq \mathbf{G}$ be the affine subgroup consisting of all automorphisms that fix pointwise the centre of L. Then $\mathbf{G}$ splits as*

$$\mathbf{G} \cong \mathbf{B}_2 \ltimes \mathbf{G}_0,$$

where, for every field extension k of $\mathbb{Q}$, the group $\mathbf{B}_2(k)$ is the group of invertible lower-triangular 2×2 matrices, and

$$\mathbf{G}_0(k) \cong \mathsf{SL}_2(R) \ltimes \mathsf{V}_{\mathrm{st}}(R)^{\oplus 2}, \quad \text{for } R = k[t]/(t^m) \text{ and } \mathsf{V}_{\mathrm{st}}(R) = R^2,$$

with respect to the standard left action. In particular, the algebraic group $\mathbf{G}$ is connected.

Remark 1.11: In fact, the description of $\mathbf{G}_0$ given in Theorem 1.10 holds true more generally, for $\mathbb{Z}$-Lie lattices corresponding to arbitrary primary polynomials; see Theorem 2.3 below. The description of the quotient of $\mathbf{G}$ by $\mathbf{G}_0$, however, becomes more involved; see [7].

The proof of Theorem 1.10, along with explicit forms of the automorphism groups, is given in Section 2. Our considerations in this context overlap somewhat with the treatment in [9]. In [7] we give a complete description of the algebraic automorphism groups of all $\mathbb{Z}$-Lie lattices associated to Grunewald–Segal representatives of $\mathbb{Q}$-indecomposable D^*-groups of even Hirsch length, based on a more technical analysis of the Lie algebras associated to (subgroups of) the algebraic automorphism groups.

1.3. Layout of the paper. In Section 2 we analyse and describe the algebraic automorphism groups of $\mathbb{Z}$-Lie lattices associated to indecomposable D^*-groups of even Hirsch length, corresponding to primary polynomials of the form $\Delta(t) = t^m$. In Section 3 we provide technical background regarding conditions on the algebraic automorphism group of a Lie ring that is needed for calculating pro-isomorphic zeta functions of groups. In Sections 4 and 5 we present calculations of the local pro-isomorphic zeta functions of the groups Γ_{t^2} and Γ_{t^3}. The former group can be dealt with in a quite straightforward manner, while the latter group is considerably more difficult to handle. From the description of the local zeta functions we draw conclusions about the analytic behaviour of the global pro-isomorphic zeta functions of Γ_{t^2} and Γ_{t^3}; again the treatment of the latter group, which forms Section 6, is more challenging and displays interesting features. In Section 7 we extend our results for the groups Γ_{t^2} and Γ_{t^3} to two infinite families of class-two nilpotent groups that result via 'base extensions' of corresponding Lie lattices.

1.4. Basic notation. We denote by $\mathbb{N}_0$ and $\mathbb{N}$ the non-negative and positive integers, respectively. For $S \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ we write $S_{\geq a} = \{x \in S \mid x \geq a\}$, and similarly for $S_{>a}$. For a prime p, we write $\mathbb{Q}_p$ for the field of p-adic numbers with $\mathbb{Z}_p$ its ring of integers. We denote the p-adic valuation of $x \in \mathbb{Q}_p$ by $v_p(x)$ and write $|x|_p = p^{-v_p(x)}$ for the p-adic absolute value. A Lie lattice over a commutative ring R with 1 is a finitely generated free R-module, equipped with a suitable Lie bracket.

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2. Automorphism groups of $\mathbb{Q}$-indecomposable D^*-Lie lattices

For any commutative ring R with 1 and any free $\mathbb{Z}$-module M, we use the notation ${}_RM = R \otimes_{\mathbb{Z}} M$ to denote the free R-module obtained by extension of scalars; if M carries extra algebraic structure that is compatible with extension of scalars, such as the structure of a Lie lattice, we employ the same notation. Thus a $\mathbb{Z}$-Lie lattice L sets up a Lie lattice scheme $R \rightsquigarrow {}_RL$. We realise the **algebraic automorphism group** $\mathbf{Aut}(L)$ of L, via a $\mathbb{Z}$-basis of L, as an affine $\mathbb{Z}$-group scheme $\mathbf{G} \leq \mathsf{GL}_d$, where $d = \dim_{\mathbb{Z}}(L)$ is the $\mathbb{Z}$-rank of L, so that, in particular,

$$\mathrm{Aut}({}_kL) \cong \mathbf{G}(k) \leq \mathsf{GL}_d(k) \quad \text{for every extension field } k \text{ of } \mathbb{Q},$$

and, thinking of GL_d as a subgroup of SL_{d+1} to make the arithmetic structure tangible,

$$\mathrm{Aut}(L) \cong \mathbf{G}(\mathbb{Z}) \quad \text{and} \quad \mathrm{Aut}({}_{\mathbb{Z}_p}L) \cong \mathbf{G}(\mathbb{Z}_p) \text{ for each prime } p,$$

with respect to the chosen basis. The automorphism groups arising in this paper come from nilpotent $\mathbb{Z}$-Lie lattices with rank-two centres and, for short, we refer to these as D^*-Lie lattices. Our aim here is to describe the algebraic automorphism groups of $\mathbb{Q}$-indecomposable D^*-Lie lattices of even $\mathbb{Z}$-rank $d = 2m + 2$ which admit a presentation suggested by [18, Thm. 6.3(b)] and associated with the primary polynomial $\Delta(t) = t^m$; compare with Section 3.1. The corresponding task for D^*-Lie lattices of odd $\mathbb{Z}$-rank has been carried out in [6]; the case of more general D^*-Lie lattices of even $\mathbb{Z}$-rank is considered in [7] (and turns out to be more involved).

We now give a detailed description, in coordinates, that is tailored also to our investigations of pro-isomorphic zeta functions. Let $m \in \mathbb{N}$ and consider the companion matrix

$$K = C(a_1, \ldots, a_m) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix} \in \mathsf{M}_m(\mathbb{Z}) \tag{2.1}$$

of a monic polynomial

$$\Delta_K = t^m - a_1 t^{m-1} - \cdots - a_{m-1} t - a_m \in \mathbb{Z}[t].$$

We consider the $\mathbb{Z}$-Lie lattice L of $\mathbb{Z}$-rank $2m+2$ with ordered $\mathbb{Z}$-basis

$$\mathcal{S} = (x_1, \ldots, x_m,\ y_1, \ldots, y_m,\ z_1, z_2)$$

and the Lie bracket defined by

$$\begin{aligned} &[x_i, y_j] = \delta_{i,j} z_1 + K_{ij} z_2, \\ &[x_i, x_j] = [y_i, y_j] = [x_i, z_1] = [x_i, z_2] = [y_i, z_1] = [y_i, z_2] = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq i, j \leq m, \end{aligned} \tag{2.2}$$

where $\delta_{i,j}$ denotes the Kronecker-delta. We observe that L is a D^*-Lie lattice with centre

$$Z = \mathrm{Z}(L) = \mathbb{Z}z_1 + \mathbb{Z}z_2.$$

Let $\mathbf{G} \leq \mathsf{GL}_{2m+2}$ be the algebraic automorphism group of L with the embedding defined by the ordered basis $\mathcal{S}$. In particular, for every integral domain k of characteristic 0, the coordinate maps with respect to $\mathcal{S}$ identify ${}_kL$ with the module k^{2m+2} of row-vectors, and the action of the group

$$\mathbf{G}(k) = \mathrm{Aut}({}_kL) \leq \mathsf{GL}_{2m+2}(k)$$

on ${}_kL$ corresponds to matrix multiplication from the right. We write $\mathbf{G}_0 \trianglelefteq \mathbf{G}$ for the affine subgroup and $\mathbb{Z}$-subscheme arising as the kernel of the natural restriction homomorphism

$$\mathbf{G}_0(k) = \mathrm{Ker}(\mathbf{G}(k) \xrightarrow{\mathrm{Res}^L_Z} \mathsf{GL}({}_kZ)). \tag{2.3}$$

From now on without further reference, let k denote an integral domain of characteristic 0. Recall that an $n \times n$ matrix over k is **regular** (or cyclic) over k, if it is similar over k to a companion matrix; such a matrix yields a

regular element of the Lie lattice $\mathsf{gl}_n(k)$, i.e., an element whose centraliser has the smallest possible rank n. The fact that the matrix K is regular plays a central role in the elucidation of $\mathbf{G}$, and it is convenient to note down two elementary facts.

Remark 2.1: Let $X, Y \in \mathsf{M}_n(k)$ be regular $n \times n$ matrices over k. Then:

(1) The centraliser of X is the polynomial algebra that it generates:

$$\mathrm{C}_{\mathsf{M}_n(k)}(X) = k[X].$$

(2) If X and Y have the same characteristic polynomial, then X and Y are similar over k.

The Lie bracket of ${}_kL$ induces an anti-symmetric bilinear map

$$[\cdot,\cdot]\colon {}_kL/{}_kZ \times {}_kL/{}_kZ \longrightarrow {}_kZ \tag{2.4}$$

with values in ${}_kZ$ which, by a slight abuse of notation, we continue to denote by $[\cdot,\cdot]$. The structure of $\mathbf{G}(k)$ is tightly connected with the symmetries of two k-valued bilinear forms on the free module $k^{2m} \cong {}_kL/{}_kZ$ that can be derived from the map described in (2.4). For any matrix $Q \in \mathsf{M}_m(\mathbb{Z})$, the matrix

$$\jmath_Q = \begin{pmatrix} 0 & Q \\ -Q^\top & 0 \end{pmatrix} \in \mathsf{M}_{2m}(\mathbb{Z})$$

can be regarded as the structure matrix of an anti-symmetric bilinear form $\langle\cdot,\cdot\rangle_{\jmath_Q}$ on k^{2m}. Let $\mathsf{O}_{\jmath_Q} \leq \mathsf{GL}_{2m}$ be the affine $\mathbb{Z}$-group scheme such that $\mathsf{O}_{\jmath_Q}(k)$ consists of all elements of $\mathsf{GL}_{2m}(k)$ that preserve the form $\langle\cdot,\cdot\rangle_{\jmath_Q}$, that is,

$$\mathsf{O}_{\jmath_Q}(k) = \{g \in \mathsf{GL}_{2m}(k) \mid g\,\jmath_Q\,g^\top = \jmath_Q\}.$$

We remark that, if $Q = \mathrm{I}_m$ is the identity matrix, the group scheme $\mathsf{O}_{\jmath_{\mathrm{I}_m}}$ is simply the classical symplectic group Sp_{2m}.

2.1. The structure of the algebraic subgroup $\mathbf{G}_0$. We start with the structure of $\mathbf{G}_0 \trianglelefteq \mathbf{G}$, the algebraic subgroup and $\mathbb{Z}$-subscheme, whose group of k-points $\mathbf{G}(k)$ fixes the centre ${}_kZ = \mathrm{Z}({}_kL)$ pointwise. An element $g \in \mathbf{G}(k) \leq \mathsf{GL}_{2m+2}(k)$ can be written as a block matrix

$$g = \begin{pmatrix} X & U \\ 0 & Y \end{pmatrix}, \tag{2.5}$$

$$\text{with } X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{GL}_2(\mathsf{M}_m(k)), \quad Y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{GL}_2(k), \quad U \in \mathsf{M}_{2m,2}(k),$$

where X and Y correspond to the automorphisms that g induces naturally on ${}_kL/{}_kZ$ and ${}_kZ$. Each of the following equivalent conditions characterises elements of $\mathbf{G}(k)$ among arbitrary elements g of the form (2.5):

$$[u,v]g = [ug, vg] \qquad \text{for all } u,v \in {}_kL;$$
$$[\bar{u},\bar{v}]Y = [\bar{u}X, \bar{v}X] \qquad \text{for all } \bar{u},\bar{v} \in {}_kL/{}_kZ;$$

$$a\mathcal{J}_{\mathrm{I}_m} + c\mathcal{J}_K = X\mathcal{J}_{\mathrm{I}_m}X^\top \quad \text{and} \quad b\mathcal{J}_{\mathrm{I}_m} + d\mathcal{J}_K = X\mathcal{J}_K X^\top. \tag{2.6}$$

From (2.6) we directly obtain a characterisation of $\mathbf{G}(k)$.

PROPOSITION 2.2: *Let $g \in \mathsf{GL}_{2m+2}(k)$ be a block matrix of the form (2.5). Then:*

(1) *$g \in \mathbf{G}(k)$ if and only if the following four conditions are satisfied:*
   (i) $BA^\top = AB^\top$ *and* $BK^\top A^\top = AKB^\top$,
   (ii) $CD^\top = DC^\top$ *and* $CKD^\top = DK^\top C^\top$,
   (iii) $a\mathrm{I}_m + cK = AD^\top - BC^\top$,
   (iv) $b\mathrm{I}_m + dK = AKD^\top - BK^\top C^\top$.

(2) *$g \in \mathbf{G}_0(k)$ if and only if $Y = \mathrm{I}_2$ and $X \in \mathsf{O}_{\mathcal{J}_{\mathrm{I}_m}}(k) \cap \mathsf{O}_{\mathcal{J}_K}(k)$, or explicitly:* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \mathrm{I}_2$ *and*
   (i) $BA^\top = AB^\top$ *and* $BK^\top A^\top = AKB^\top$,
   (ii) $CD^\top = DC^\top$ *and* $CKD^\top = DK^\top C^\top$,
   (iii)$_0$ $\mathrm{I}_m = AD^\top - BC^\top$,
   (iv)$_0$ $K = AKD^\top - BK^\top C^\top$.

The proof of the following key theorem was inspired by a more technical analysis of the Lie algebras associated to subgroups of $\mathbf{G}$, carried out in [7], and by-passes the use of Lie algebras by means of a computational trick.

THEOREM 2.3: *The affine group scheme $\mathbf{G}_0$ splits as follows:*

$$\mathbf{G}_0(k) \cong \mathsf{SL}_2(k[K]) \ltimes \mathsf{V}_{\mathrm{st}}(k[K])^{\oplus 2},$$

where $\mathsf{V}_{\mathrm{st}}(\cdot)$ denotes the standard left $\mathsf{SL}_2(\cdot)$-module.

Proof. Recall that every square matrix over a field is similar to its transpose and that the conjugating matrix may be taken to be symmetric. In fact, for regular matrices it is always symmetric; compare with [34]. Therefore, there exists a symmetric matrix $\sigma \in \mathsf{GL}_m(\mathbb{Q})$ such that $K^\top = \sigma K \sigma^{-1}$. In our special situation, we can even arrange that $\sigma \in \mathsf{GL}_m(\mathbb{Z})$, because the factor groups $\mathbb{Z}^m$

modulo the row-span of K and $\mathbb{Z}^m$ modulo the column-span of K are isomorphic (cyclic) groups. We set

$$\Sigma = \begin{pmatrix} \mathrm{I}_m & \\ & \sigma \end{pmatrix} \in \mathsf{GL}_{2m}(\mathbb{Z}), \quad \text{where } K^\top = \sigma K \sigma^{-1}, \tag{2.7}$$

and claim that for every $g \in \mathsf{GL}_{2m+2}(k)$ of the form (2.5), with $Y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \mathrm{I}_2$, the following holds:

$$g \in \mathbf{G}_0(k) \quad \text{if and only if} \quad \Sigma^{-1} X \Sigma \in \mathsf{SL}_2(k[K]). \tag{2.8}$$

First suppose that $g \in \mathbf{G}_0(k)$. By Proposition 2.2 (2), this implies that $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{GL}_2(\mathsf{M}_m(k))$ satisfies conditions (i)–(iv)$_0$. From (i), (ii), (iii)$_0$ and (iii)$_0^\top$—the transpose of (iii)$_0$—we obtain

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{pmatrix}. \tag{2.9}$$

Now, using the fact that the inverse $g^{-1} \in \mathbf{G}_0(k)$ satisfies a similar set of equations, we get

(i)$'$ $B^\top D = D^\top B$ and $B^\top K^\top D = D^\top K B$,
(ii)$'$ $C^\top A = A^\top C$ and $C^\top K A = A^\top K^\top C$,
(iii)$'_0$ $\mathrm{I}_m = D^\top A - B^\top C$,
(iv)$'_0$ $K = D^\top K A - B^\top K^\top C$.

Using these additional conditions we deduce that

$$AK = KA, \quad KB = BK^\top, \quad CK = K^\top C, \quad K^\top D = DK^\top. \tag{2.10}$$

Indeed, multiplying (iv)$'_0$ by A on the left gives

$$\begin{aligned} AK &= AD^\top KA - AB^\top K^\top C \overset{\text{(iii)}_0}{=} (\mathrm{I}_m + BC^\top)KA - AB^\top K^\top C \\ &\overset{\text{(ii)}'}{=} KA + BA^\top K^\top C - AB^\top K^\top C \overset{\text{(i)}}{=} KA; \end{aligned}$$

multiplying (iv)$'_0$ by $B^\top$ on the right gives

$$\begin{aligned} KB^\top &= D^\top KAB^\top - B^\top K^\top CB^\top \overset{\text{(iii)}_0^\top}{=} D^\top KAB^\top - B^\top K^\top (DA^\top - \mathrm{I}_m) \\ &\overset{\text{(i)}}{=} D^\top KBA^\top - B^\top K^\top DA^\top + B^\top K^\top \overset{\text{(i)}'}{=} B^\top K^\top; \end{aligned}$$

multiplying $(\mathrm{iv})'_0$ by C on the left gives

$$\begin{aligned} CK &= CD^\top KA - CB^\top K^\top C \overset{(\mathrm{iii})_0^\top}{=} CD^\top KA - (DA^\top - \mathrm{I}_m)K^\top C \\ &\overset{(\mathrm{ii})}{=} DC^\top KA - DA^\top K^\top C + K^\top C \overset{(\mathrm{ii})'}{=} K^\top C; \end{aligned}$$

and multiplying $(\mathrm{iv})'_0$ by $D^\top$ on the right gives

$$\begin{aligned} KD^\top &= D^\top KAD^\top - B^\top K^\top CD^\top \overset{(\mathrm{iii})_0}{=} D^\top K(\mathrm{I}_m + BC^\top) - B^\top K^\top CD^\top \\ &\overset{(\mathrm{ii})}{=} D^\top K + D^\top KBC^\top - B^\top K^\top DC^\top \overset{(\mathrm{i})'}{=} D^\top K. \end{aligned}$$

Recalling the definition of Σ in (2.7) and rewriting the relations (2.10), we get

$$\begin{gathered} AK = KA, \qquad K(B\sigma) = (B\sigma)K, \\ (\sigma^{-1}C)K = K(\sigma^{-1}C), \quad (\sigma^{-1}D\sigma)K = K(\sigma^{-1}D\sigma). \end{gathered}$$

By Remark 2.1, this implies that $A, B\sigma, \sigma^{-1}C, \sigma^{-1}D\sigma \in k[K]$, that is,

$$\Sigma^{-1}X\Sigma = \begin{pmatrix} A & B\sigma \\ \sigma^{-1}C & \sigma^{-1}D\sigma \end{pmatrix} \in \mathsf{GL}_2(k[K]).$$

From $B\sigma, \sigma^{-1}C \in k[K]$ and the symmetry of σ, one readily obtains that B, C are symmetric. From $(\mathrm{iii})_0$ and $\sigma^{-1}D\sigma = D^\top$ we obtain that $\Sigma^{-1}X\Sigma \in \mathsf{SL}_2(k[K])$.

Conversely, suppose that

$$\begin{pmatrix} A & B\sigma \\ \sigma^{-1}C & \sigma^{-1}D\sigma \end{pmatrix} = \Sigma^{-1}X\Sigma \in \mathsf{SL}_2(k[K]).$$

It suffices to check the conditions (i)–$(\mathrm{iv})_0$ in Proposition 2.2 (2). This can be done by routine computations, using $K^\top = \sigma K \sigma^{-1}$ and the fact that $k[K]$ is commutative. For instance, from $\sigma^{-1}D\sigma, \sigma^{-1}C \in k[K]$ and $\sigma^\top = \sigma$ we obtain

$$D = \sigma(\sigma^{-1}D\sigma)\sigma^{-1} = (\sigma^{-1}D\sigma)^\top = \sigma D^\top \sigma^{-1},$$

thus $\sigma^{-1}D\sigma = D^\top$, and

$$C\sigma^{-1} = \sigma(\sigma^{-1}C)\sigma^{-1} = (\sigma^{-1}C)^\top = C^\top \sigma^{-1},$$

thus $C = C^\top$. This yields

$$\mathrm{I}_m = \det \begin{pmatrix} A & B\sigma \\ \sigma^{-1}C & \sigma^{-1}D\sigma \end{pmatrix} = A \cdot \sigma^{-1}D\sigma - B\sigma \cdot \sigma^{-1}C = AD^\top - BC^\top,$$

and $(\mathrm{iii})_0$ holds. This concludes the justification of (2.8).

Finally, the block matrix $U \in \mathsf{M}_{2m,2}(k)$ in (2.5) remains unconstrained in Proposition 2.2 and therefore the group is isomorphic to $\mathsf{SL}_2(k[K]) \ltimes \mathsf{M}_{2m,2}(k)$. We can identify the natural $k[K]$-module k^m with the standard $k[K]$-module $k[K]$, by mapping a cyclic generator of k^m to the cyclic generator K of $k[K]$. Therefore $\mathsf{M}_{2m,2}(k)$ can be replaced by a direct sum of two copies of the standard $\mathsf{SL}_2(k[K])$-module $\mathsf{V}_{\mathrm{st}}(k[K])$. ∎

2.2. The structure of the algebraic automorphism group $\mathbf{G}$ for $\Delta_K = t^m$. Now we focus on the special case $\Delta_K = t^m$; that is, the case

$$K = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathsf{M}_m(\mathbb{Z}). \tag{2.11}$$

In this situation we can take

$$\sigma = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathsf{GL}_m(\mathbb{Z}), \quad \text{and} \quad \Sigma = \begin{pmatrix} \mathrm{I}_m & \\ & \sigma \end{pmatrix} \in \mathsf{GL}_{2m}(\mathbb{Z}) \tag{2.12}$$

in the analysis carried out in Section 2.1. We remark that this particular choice of σ corresponds to the longest element in the symmetric group $\mathrm{Sym}(m)$, with respect to the standard generators as a Coexeter group.

Proposition 2.4: *Suppose that K has characteristic polynomial $\Delta_K = t^m$. Then the natural restriction homomorphism* (2.3) *sets up, over $\mathbb{Z}$, a split short exact sequence*

$$\mathbf{G}_0(k) \hookrightarrow \mathbf{G}(k) \xrightarrow{\mathrm{Res}} \underbrace{\mathbf{B}_2(k)}_{\cong \mathbf{G}(k)/\mathbf{G}_0(k)} \leq \mathsf{GL}_2(k),$$

where $\mathbf{B}_2(k)$ is the group of invertible lower-triangular 2×2 matrices.

Proof. We show below that the image of $\mathbf{G}(k)$ in $\mathsf{GL}_2(k)$ under the restriction homomorphism

(a) contains $\mathbf{B}_2(k)$ by exhibiting an explicit section over $\mathbb{Z}$, but
(b) does not contain elements of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \neq 0$.

From this it follows that the image is precisely $\mathbf{B}_2(k)$, because, once we replace k by its field of fractions, there are no properly intermediate subgroups between $\mathbf{B}_2(k)$ and $\mathsf{GL}_2(k)$, as can be seen from the Bruhat decomposition.

To prove (a), we define for $a, d \in k^\times$ and $c \in k$ the following elements of $\mathsf{GL}_{2m+2}(k)$:

$$\begin{aligned} U(a) &= \operatorname{diag}(a, a^2, \dots, a^m, 1, a^{-1}, \dots, a^{-m+1}, a, 1), \\ V(d) &= \operatorname{diag}(d^{-1}, d^{-2}, \dots, d^{-m}, d, d^2, \dots, d^m, 1, d), \\ W(c) &= \operatorname{diag}\left(\exp(cE_m), \exp(cE_m^\vee), \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \right), \end{aligned} \tag{2.13}$$

where $\exp(t) = \sum_{n=0}^{\infty} t^n/(n!)$ denotes the exponential series (which, evaluated on nilpotent $m \times m$-matrices, can be truncated after the mth term and thus produces finite sums) and

$$E_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & 0 \\ 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & (m-1) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$E_m^\vee = \begin{pmatrix} 0 & 0 \\ 0 & -{E_{m-1}}^\top \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ 0 & -1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -(m-2) & 0 \end{pmatrix}.$$

A direct calculation reveals that the elements $U(a), V(d)$ and $W(c)$ satisfy (iii) and (iv) of Proposition 2.2, while (i) and (ii) hold trivially; thus $U(a), V(d), W(c) \in \mathbf{G}(k)$. Moreover, there is an affine subgroup and $\mathbb{Z}$-subscheme $\mathbf{B} \leq \mathbf{G}$ such that

$$\mathbf{B}(k) = \{U(a)V(d)W(c) \mid a, d \in k^\times \text{ and } c \in k\} \quad \text{and} \quad \mathbf{B}(k) \cong \mathbf{B}_2(k)$$

via the natural restriction homomorphism, which satisfies

$$U(a) \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad V(d) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad W(c) \mapsto \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix};$$

the inverse can be built from the morphisms $a \mapsto U(a)$, $d \mapsto V(d)$ and $c \mapsto W(c)$ which are defined over $\mathbb{Z}$. The latter is clear for $U(\cdot)$ and $V(\cdot)$, and requires a routine calculation for $W(\cdot)$: by induction, one sees that the factorials in the denominators coming from the exponential series duly cancel out with the entries of the relevant powers of cE_m and $cE_m^\vee$.

To prove (b) we observe that $(\mathrm{iii})_0'$ in the proof of Theorem 2.3 holds also for elements $g \in \mathbf{G}(k)$ of the form (2.5) which satisfy $Y = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. Taking the trace in equation (iv) of Proposition 2.2 (2), we obtain

$$\begin{aligned} &mb + \mathrm{tr}(K) \\ &\quad = \mathrm{tr}(b\mathrm{I}_m + K) = \mathrm{tr}(AKD^\top - BK^\top C^\top) && \text{by taking the trace in (iv)} \\ &\quad = \mathrm{tr}(KD^\top A - C^\top BK^\top) && \text{by permuting matrices} \\ &\quad = \mathrm{tr}(KD^\top A - KB^\top C) && \text{by transposing the second matrix} \\ &\quad = \mathrm{tr}(K) && \text{by applying } (\mathrm{iii})_0', \end{aligned}$$

and this implies $b = 0$.

We remark that, alternatively, one can prove (b) as follows. Every $g \in \mathbf{G}(k)$ restricts to an automorphism of the centre ${}_kZ$ of ${}_kL$, which is represented by $Y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{GL}_2(k)$ with respect to the chosen basis z_1, z_2, and similarly for g^{-1}. The image (z_1', z_2') of the pair (z_1, z_2) under g^{-1} yields two anti-symmetric bilinear forms which encode the Lie bracket; inspection of the form associated to z_2' shows that $b\mathrm{I}_m + dK$ should have the same rank as K, namely $m - 1$; thus $b = 0$. ∎

Proof of Theorem 1.10. In view of Theorem 2.3 and Proposition 2.4, it only remains to show that the algebraic group $\mathbf{G}$ is connected. As

$$\mathbf{G}(k) \cong \mathbf{B}_2(k) \ltimes (\mathsf{SL}_2(k[K]) \ltimes \mathsf{V}_{\mathrm{st}}(k[K])^{\oplus 2})$$

by an isomorphism of group schemes over $\mathbb{Z}$, the connectedness of $\mathbf{G}$ follows from the fact that $\mathbf{G}$ is generated by one-parameter subgroups, which are, in particular, affine irreducible varieties containing 1; for instance, see [27, Prop. 1.16]. ∎

For our next step we record also the following consequence of Theorem 2.3 and Proposition 2.4.

COROLLARY 2.5: *Suppose that K has characteristic polynomial $\Delta_K = t^m$. Then the group $\mathbf{G}_0(k)$ is conjugate to the subgroup of $\mathsf{GL}_{2m+2}(k)$ consisting of elements of the form*

$$\begin{pmatrix} A & B & E \\ C & D & F \\ 0 & 0 & \mathrm{I}_2 \end{pmatrix},$$

where $A, B, C, D \in \mathsf{M}_m(k)$ satisfy $AD - BC = \mathrm{I}_m$ and are in Toeplitz form, that is,

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ & a_1 & \ddots & \vdots \\ & & \ddots & a_2 \\ & & & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 & \cdots & b_m \\ & b_1 & \ddots & \vdots \\ & & \ddots & b_2 \\ & & & b_1 \end{pmatrix},$$

$$C = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ & c_1 & \ddots & \vdots \\ & & \ddots & c_2 \\ & & & c_1 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & d_2 & \cdots & d_m \\ & d_1 & \ddots & \vdots \\ & & \ddots & d_2 \\ & & & d_1 \end{pmatrix} \tag{2.14}$$

with suitable entries $a_1, \ldots, d_m \in k$ and 0 in white spaces, and $E, F \in \mathsf{M}_{m,2}(k)$.

The group $\mathbf{G}(k)$ is generated by $\mathbf{G}_0(k)$ and the elements $U(a)$, $V(d)$ and $W(c)$, for $a, d \in k^\times$ and $c \in k$, which are defined in the proof of Proposition 2.4.

2.3. CHANGE OF COORDINATES. For $\Delta_K = t^m$, the Lie lattice L is intimately linked to the nilpotent group Γ_{t^m}, defined in (1.3), and the algebraic automorphism group $\mathbf{G}$ plays a central role in the treatment of the pro-isomorphic zeta function of Γ_{t^m}; see Section 3. With a view towards the computation of the pro-isomorphic zeta function of the group Γ_{t^m}, we perform a change of basis from

$$\mathcal{S} = (x_1, x_2, \ldots, x_m,\ y_1, y_2, \ldots, y_m,\ z_1, z_2)$$

to

$$\mathcal{S}^* = (x_1, y_m,\ x_2, y_{m-1},\ \ldots,\ x_m, y_1,\ z_2, z_1).$$

This basis change is achieved by conjugating first with $\mathrm{diag}(\Sigma, \mathrm{I}_2)$, already built into Corollary 2.5 and reversing the order of $y_1, \ldots, y_m$, and then

with $\mathrm{diag}(\Theta, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, where Θ corresponds to the permutation of $\{1, 2, \ldots, 2m\}$ given by

$$\begin{cases} i \mapsto 2i - 1 & \text{if } 1 \leq i \leq m, \\ i \mapsto 2(i-m) & \text{if } m < i \leq 2m. \end{cases} \tag{2.15}$$

From the results in Section 2.2 we obtain the following description of $\mathbf{G}(k)$, with respect to the basis $\mathcal{S}^*$.

PROPOSITION 2.6: *Suppose that K has characteristic polynomial $\Delta_K = t^m$. Then, with respect to the basis $\mathcal{S}^*$, the elements of $\mathbf{G}_0(k)$ take the form*

$$\left(\begin{array}{ccccc|cc} X_1 & X_2 & X_3 & \cdots & X_m & * & * \\ & X_1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & \ddots & \ddots & X_3 & * & * \\ & & & X_1 & X_2 & * & * \\ & & & & X_1 & * & * \\ \hline & & & & & 1 & 0 \\ & & & & & & 1 \end{array}\right), \tag{2.16}$$

with $X_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \mathsf{M}_2(k)$ for $1 \leq i \leq m$ and arbitrary entries in the positions marked $*$,

such that the matrices A, B, C, D defined as in (2.14) satisfy $AD - BC = \mathrm{I}_m$.

Furthermore, still with respect to the basis $\mathcal{S}^*$, the group $\mathbf{G}(k)$ is generated by $\mathbf{G}_0(k)$ and

$$\begin{aligned} U'(a) &= T^{-1}(U(a)V(a))T = \mathrm{diag}\left(\begin{pmatrix} 1 & \\ & a \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \\ & a \end{pmatrix}, \begin{pmatrix} a & \\ & a \end{pmatrix}\right), \\ V'(d) &= T^{-1}(U(d)^{m-1}V(d)^m R(d)^m)\, T \\ &= \mathrm{diag}\left(d^{m-1}\mathrm{I}_2, d^{m-2}\mathrm{I}_2, \ldots, d\mathrm{I}_2, \mathrm{I}_2, \begin{pmatrix} d^m & \\ & d^{m-1} \end{pmatrix}\right), \\ W'(c) &= T^{-1}W(c)T \end{aligned} \tag{2.17}$$

for $a, d \in k^\times$ and $c \in k$, where $T = \mathrm{diag}(\Sigma\Theta, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, the one-parameter groups $U(\cdot), V(\cdot), W(\cdot)$ are as in (2.13) and

$$R(d) = \mathrm{diag}(d, d, \ldots, d, d^{-1}, d^{-1}, \ldots, d^{-1}, 1, 1) \in \mathbf{G}_0(k).$$

COROLLARY 2.7: *Suppose that K has characteristic polynomial $\Delta_K = t^m$. Then the quotient of $\mathbf{G}$ by its unipotent radical $\mathbf{N}$ is isomorphic to $\mathsf{GL}_2 \times \mathsf{GL}_1$, with an explicit section defined over $\mathbb{Z}$ with respect to the basis $\mathcal{S}^*$ as follows:*

$\mathsf{GL}_2(k) \times \mathsf{GL}_1(k) \to \mathbf{H}(k)$,

$$(A, \nu) \mapsto \left(\begin{array}{cccc|cc} \nu^{m-1}A & & & 0 & & \\ & \nu^{m-2}A & & & & \\ & & \ddots & & & \\ & & & \nu A & & \\ 0 & & & A & & \\ \hline & & & & \nu^m \det A & 0 \\ & & & & 0 & \nu^{m-1} \det A \end{array}\right).$$

Proof. We consider the affine subgroup $\mathbf{N}$ of $\mathbf{G}$ such that $\mathbf{N}(k)$ is generated by elements of the form (2.16) with $X_1 = \mathrm{I}_2$ together with elements of the subgroup $\{W'(c) \mid c \in k\}$: see (2.17). The group $\mathbf{N}$ is a connected unipotent normal subgroup of $\mathbf{G}$.

Moreover, the quotient $\mathbf{G}(k)/\mathbf{N}(k)$ is generated by the block-diagonal matrices $\mathrm{diag}(X_1, \dots, X_1, \mathrm{I}_2)$ with $X_1 \in \mathsf{SL}_2(k)$ and by the one-parameter subgroups $\{U'(a) \mid a \in k^\times\}$ and $\{V'(\nu) \mid \nu \in k^\times\}$; this analysis also provides a section for $\mathbf{G} \to \mathbf{G}/\mathbf{N}$ over $\mathbb{Z}$. Finally $\mathbf{G}/\mathbf{N} \cong \mathsf{GL}_2 \times \mathsf{GL}_1$ is reductive, and thus $\mathbf{N}$ is the unipotent radical of $\mathbf{G}$. ∎

Remark 2.8: For computational purposes we replaced the generators $U(\cdot)$ and $V(\cdot)$ by the generators $U'(\cdot)$ and $V'(\cdot)$. They generate the same torus, modulo $\mathbf{G}_0$ and up to coordinate change; see (2.17).

For similar reasons, a further simplification of the computation of the pro-isomorphic zeta function can be achieved by replacing the one-parameter subgroup $W'(\cdot)$ in (2.17) by $c \mapsto W''(c)$, where

$$W''(c) = T^{-1}\mathrm{diag}(\exp(cE_m + \tfrac{1}{2}(1-m)cK), \exp(cE_m^\vee - \tfrac{1}{2}(1-m)cK^\top), \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right))T.$$

This switch is inspired by Lie algebra considerations and works for an arbitrary $\mathbb{Z}$-algebra k if m is odd; for even m the switch requires that k is a $\mathbb{Z}[\frac{1}{2}]$-algebra. For the applications in the present paper, we do not need the variant for $m = 2$ and we use it only for $m = 3$. Hence no primes need to be excluded when we compute the local pro-isomorphic zeta functions for Theorems 1.1 and 1.3.

Example 2.9: In order to compute later on the pro-isomorphic zeta functions of the groups Γ_{t^m} for $m \in \{2,3\}$, we record in these cases explicit descriptions of the unipotent radical $\mathbf{N}$ of $\mathbf{G}$, with respect to the basis $\mathcal{S}^*$. For completeness we also provide a description for $m = 1$ which is straightforward; compare with [3, §3.3.4]. We have

$$\mathbf{N}(k) = \left\{ \left(\begin{array}{cc|cc} \mathrm{I}_2 & & * & * \\ \hline & & 1 & \lambda \\ & & & 1 \end{array} \right) \mid \lambda \in k, \text{ and arbitrary entries in the positions marked } * \right\} \text{ if } m=1,$$

$$\mathbf{N}(k) = \left\{ \left(\begin{array}{cc|cc} \mathrm{I}_2 & X_2 & * & * \\ 0 & \mathrm{I}_2 & * & * \\ \hline & & 1 & \mathrm{tr}(X_2) \\ & & & 1 \end{array} \right) \mid X_2 \in \mathsf{M}_2(k), \text{ and arbitrary entries in the positions marked } * \right\} \text{ if } m=2,$$

$$\mathbf{N}(k) = \left\{ \left(\begin{array}{ccc|cc} \mathrm{I}_2 & X_2 & X_3 & * & * \\ 0 & \mathrm{I}_2 & X_2 + \lambda \mathrm{I}_2 & * & * \\ 0 & 0 & \mathrm{I}_2 & * & * \\ \hline & & & 1 & \lambda \\ & & & & 1 \end{array} \right) \mid \begin{array}{l} X_2, X_3 \in \mathsf{M}_2(k) \text{ with } \mathrm{tr}(X_2)=0, \\ \mathrm{tr}(X_3) + \det(X_2) = 0,\ \lambda \in k, \\ \text{and arbitrary entries} \\ \text{in the positions marked } * \end{array} \right\}$$

if $m=3$.

Indeed, for $m = 2$ we substitute $X_1 = \mathrm{I}_2$ in (2.16) and use the determinant equation in Proposition 2.6 to obtain

$$\begin{pmatrix} a_1 & a_2 \\ & a_1 \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ & d_1 \end{pmatrix} - \begin{pmatrix} b_1 & b_2 \\ & b_1 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ & c_1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

with $a_1 - 1 = d_1 - 1 = b_1 = c_1 = 0$, and therefore $a_2 + d_2 = 0$, namely $\mathrm{tr}(X_2) = 0$; this accounts for the contribution of $\mathbf{G}_0(k)$. The explicit form of $W'(c)$ for $c \in k$ is

$$W'(c) = \left(\begin{array}{cccc|cc} 1 & 0 & c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ \hline & & & & 1 & c \\ & & & & 0 & 1 \end{array} \right).$$

Combining the contributions, the result for $m = 2$ follows.

For $m = 3$, we start again by substituting $X_1 = \mathrm{I}_2$ in (2.16), which together with the determinant equation

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ & a_1 & a_2 \\ & & a_1 \end{pmatrix} \begin{pmatrix} d_1 & d_2 & d_3 \\ & d_1 & d_2 \\ & & d_1 \end{pmatrix} - \begin{pmatrix} b_1 & b_2 & b_3 \\ & b_1 & b_2 \\ & & b_1 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \\ & c_1 & c_2 \\ & & c_1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

gives $a_1 - 1 = d_1 - 1 = b_1 = c_1 = 0$ and

$$\operatorname{tr}(X_2) = a_2 + b_2 = 0,$$

$$\operatorname{tr}(X_3) + \det(X_3) = a_3 + d_3 + a_2 d_2 - b_2 c_2 = 0;$$

this yields the intersection of the unipotent radical $\mathbf{N}(k)$ with $\mathbf{G}_0(k)$.

Offsetting $W'(\cdot)$ in accordance with Remark 2.8, we get the one-parameter subgroup $W''(\cdot)$ which takes the form

$$W''(c) = \left(\begin{array}{ccc|cc} \mathrm{I}_2 & 0 & 0 & & \\ 0 & \mathrm{I}_2 & c\mathrm{I}_2 & & \\ 0 & 0 & \mathrm{I}_2 & & \\ \hline & & & 1 & c \\ & & & & 1 \end{array}\right), \quad \text{for } c \in k.$$

Combining the contributions, we arrive at the result for $m = 3$.

3. Machinery for computing p-adic integrals over algebraic groups

In this section we collect various facts and notation in order to use the technology developed in [19, 23, 14, 4]. The general treatment produces a finite, but typically unspecified set of 'exceptional' primes; we take care to verify that, for the applications in this paper, there is no need to exclude any primes.

3.1. Lie correspondence for class-two nilpotent groups. Let Γ be a finitely generated torsion-free nilpotent group. Grunewald, Segal and Smith [19, Thm. 4.1] showed that the local pro-isomorphic zeta functions of Γ are closely linked to the local pro-isomorphic zeta functions of a nilpotent $\mathbb{Z}$-Lie lattice L that can be constructed from Γ; indeed,

$$\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s)$$

for almost all primes p. Furthermore, they remark that, if Γ has nilpotency class two, a suitable Lie correspondence can be implemented more directly, and they highlight consequences for other types of zeta functions. The direct correspondence has been reinterpreted and put to use, for instance, in [32, §2.4.1]. For the record, we state and explain the implications of the special construction in nilpotency class two for pro-isomorphic zeta functions, where it is applied not merely to a group, but also to its lattice of subgroups; compare with [6, Rem. 2.2].

Let Γ be a finitely generated torsion-free class-two nilpotent group of Hirsch length d, and let $Z = \mathrm{Z}(\Gamma)$ denote its centre. Then the isomorphism type of Γ is uniquely determined by

$$\Gamma/Z = \langle x_1 Z, \ldots, x_a Z \rangle \cong \mathbb{Z}^a, \quad Z = \langle y_1, \ldots, y_{d-a} \rangle \cong \mathbb{Z}^{d-a}$$

and the map

$$\gamma \colon \Gamma/Z \times \Gamma/Z \to Z, \quad (gZ, hZ) \mapsto [g, h].$$

In fact, this data yields a $\mathbb{Z}$-Lie lattice

$$L = \mathbb{Z}\dot{x}_1 \oplus \cdots \oplus \mathbb{Z}\dot{x}_a \oplus \mathbb{Z}\dot{y}_1 \oplus \cdots \oplus \mathbb{Z}\dot{y}_{d-a} \cong \Gamma/Z \oplus Z, \tag{3.1}$$

where the Lie bracket is induced by the anti-symmetric bi-additive map γ and the stipulation that $\mathbb{Z}\dot{y}_1 \oplus \cdots \oplus \mathbb{Z}\dot{y}_{d-a}$ be central in L:

$$[\dot{x}_i, \dot{x}_j]_{\mathrm{Lie}} = \sum_{k=1}^{d-a} c_{i,j,k}\, \dot{y}_k \qquad \text{for } 1 \le i, j \le a, \text{ where } \gamma(x_i Z, x_j Z) = [x_i, x_j] = \prod_{k=1}^{d-a} y_k^{c_{i,j,k}};$$

$$[\dot{x}_i, \dot{y}_j]_{\mathrm{Lie}} = [\dot{y}_j, \dot{y}_k]_{\mathrm{Lie}} = 0 \quad \text{for } 1 \le i \le a \text{ and } 1 \le j \le k \le d-a.$$

Conversely, given such a Lie lattice one can define a class-two nilpotent group, essentially by factoring out from the free class-two nilpotent group on d generators $\hat{x}_1, \ldots, \hat{x}_a, \hat{y}_1, \ldots, \hat{y}_{d-a}$ the relations

$$[\hat{x}_i, \hat{x}_j]_{\mathrm{Lie}} = \prod_{k=1}^{d-a} \hat{y}_k^{c_{i,j,k}} \qquad \text{for } 1 \le i, j \le a, \text{ where } [\dot{x}_i, \dot{x}_j]_{\mathrm{Lie}} = \sum_{k=1}^{d-a} c_{i,j,k}\, y_k,$$

$$[\hat{x}_i, \hat{y}_j] = [\hat{y}_j, \hat{y}_k] = 1 \quad \text{for } 1 \le i \le a \text{ and } 1 \le j \le k \le d-a.$$

Moreover, the two constructions set up a 1-to-1 correspondence, up to isomorphism, between finitely generated torsion-free class-two nilpotent groups of Hirsch length d and class-two nilpotent $\mathbb{Z}$-Lie lattices of dimension d. For short, we call this the **class-two Lie correspondence**.

We observe that, for any prime p, essentially the same constructions yield a 'local' class-two Lie correspondence, up to isomorphism, between torsion-free class-two nilpotent pro-p groups of rank d and class-two nilpotent $\mathbb{Z}_p$-Lie lattices of dimension d; compare with [18, §1] and [32, §2.4.1].

PROPOSITION 3.1: *Let Γ be a finitely generated torsion-free class-two nilpotent group of Hirsch length d, with centre $Z = \mathrm{Z}(\Gamma)$, such that*

$$\Gamma/Z = \langle x_1 Z, \ldots, x_a Z \rangle \cong \mathbb{Z}^a \quad \text{and} \quad Z = \langle y_1, \ldots, y_{d-a} \rangle \cong \mathbb{Z}^{d-a}.$$

Let L be the $\mathbb{Z}$-Lie lattice associated to Γ under the class-two Lie correspondence as in (3.1).

Then there is an index-preserving 1*-to-*1 *correspondence between finite-index subgroups $\Delta \leq \Gamma$ and finite-index Lie sublattices $M \leq L$. Furthermore, subgroups Δ satisfying $\widehat{\Delta} \cong \widehat{\Gamma}$ are bijectively matched with Lie sublattices M such that the $\mathbb{Z}_p$-Lie lattices $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$ and $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$ are isomorphic for all primes p. In particular, this implies that*

$$\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s) = \zeta^{\mathrm{iso}}_{L_p}(s) \quad \text{for all primes } p.$$

Proof. It was already remarked in [19], just after the proof of Theorem 4.1 in that paper, that there exists an index-preserving 1-to-1 correspondence between finite-index subgroups of Γ and finite-index Lie sublattices of L.

Indeed, we can regard L as a graded $\mathbb{Z}$-Lie lattice with respect to the decomposition $L = L_{(1)} \oplus L_{(2)}$, where $L_{(1)} = \Gamma/Z$ and $L_{(2)} = Z$. Then there is a canonical index-preserving map from the set of finite-index subgroups Δ of Γ to the set of finite-index *graded* Lie sublattices of L, with finite fibers; it maps $\Delta \leq \Gamma$ to the graded Lie sublattice $M_{(1)} \oplus M_{(2)} \leq L$, where $M_{(1)} = \Delta Z/Z$, $M_{(2)} = \Delta \cap Z$, and the fiber above $M_{(1)} \oplus M_{(2)}$ has size $|L_{(2)} : M_{(2)}|^a = |Z : \Delta \cap Z|^a$. Similarly, there is a canonical index-preserving map from the set of all finite-index Lie sublattices M of L to the set of finite-index graded Lie sublattices of L, with finite fibers; it maps M to the graded Lie sublattice $M_{(1)} \oplus M_{(2)} \leq L$, where $M_{(1)}$ is the image of M under the projection from L to $L_{(1)}$ along $L_{(2)}$, $M_{(2)} = M \cap L_{(2)}$, and the fiber above $M_{(1)} \oplus M_{(2)}$ again has size $|L_{(2)} : M_{(2)}|^a$. Thus we obtain a (non-canonical) index-preserving 1-to-1 correspondence between finite-index subgroups of Γ and finite-index Lie sublattices of L, simply by matching the members of equal-sized fibres above graded Lie sublattices of L.

Next we fix a prime p, a finite-index subgroup $\Delta \leq \Gamma$ and its image $M \leq L$ under the 1-by-1 correspondence set up above so that Δ and M give rise to the same graded Lie sublattice M_{gr} of L. It remains to justify that $\widehat{\Delta}_p \cong \widehat{\Gamma}_p$ if and only if $\mathbb{Z}_p \otimes M \cong L_p$. First we observe that $\mathrm{C}_\Gamma(\Delta) = Z$ and thus $\Delta \cap Z = \mathrm{Z}(\Delta)$. This implies that M_{gr} is isomorphic to the $\mathbb{Z}$-Lie lattice associated canonically to Δ via the class-two Lie correspondence. Furthermore, M_{gr} and M are clearly isomorphic as $\mathbb{Z}$-Lie lattices. Since the constructions that lead to the class-two Lie correspondences for discrete nilpotent groups and for nilpotent pro-p groups are essentially the same, we see that the $\mathbb{Z}_p$-Lie lattice associated canonically to the pro-p completion $\widehat{\Delta}_p$ can be obtained from M by extension of scalars, i.e., it is isomorphic to $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$. The same analysis applies, of course, also to Γ in place of Δ. Applying the local class-two Lie correspondence, we deduce that $\widehat{\Delta}_p \cong \widehat{\Gamma}_p$ if and only if $\mathbb{Z}_p \otimes M \cong L_p$. ∎

Remark 3.2: In a preliminary version of this article we erroneously stated that there was a simple map $\Gamma \to L$ that would induce a 1-to-1 correspondence between finite-index subgroups of Γ and finite-index Lie sublattices of L with the properties described in Proposition 3.1. Our misunderstanding was noticed by Hyodo, who discusses the correspondence in more detail in [22, §7]. For completeness, we recall that for odd primes p, there is, in fact, a more conceptual way to establish the equalities

$$\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s) = \zeta^{\mathrm{iso}}_{L_p}(s)$$

which results from the Lie theory for torsion-free pro-p groups that are PF; compare with [17].

3.2. Local pro-isomorphic zeta functions as integrals over reductive groups. Recall from Section 2 the notion of the algebraic automorphism group $\mathbf{Aut}(L)$ of a $\mathbb{Z}$-Lie lattice L; via a $\mathbb{Z}$-basis of L, the group $\mathbf{Aut}(L)$ is realised as an affine $\mathbb{Z}$-group scheme $\mathbf{G} \leq \mathsf{GL}_d$, where d is the $\mathbb{Z}$-rank of L. As before, for any commutative ring R with 1 we write ${}_RL = R \otimes_{\mathbb{Z}} L$ and, for short, we set

$$L_p = {}_{\mathbb{Z}_p}L \quad \text{for every prime } p.$$

PROPOSITION 3.3 (Grunewald, Segal, Smith [19, Prop. 3.4]): *Let L be a nilpotent $\mathbb{Z}$-Lie lattice of $\mathbb{Z}$-rank d, and let $\mathbf{G} = \mathbf{Aut}(L) \leq \mathsf{GL}_d$ denote the algebraic automorphism group of L with respect to some $\mathbb{Z}$-basis. For each prime p, let*

$$G_p = \mathbf{G}(\mathbb{Q}_p) \quad \text{and} \quad G_p^+ = G_p \cap \mathsf{M}_d(\mathbb{Z}_p) \cong \mathrm{Aut}(_{\mathbb{Q}_p}L) \cap \mathrm{End}(_{\mathbb{Z}_p}L),$$

equipped with the right Haar measure μ_{G_p} on the locally compact group G_p such that $\mu_p(\mathbf{G}(\mathbb{Z}_p)) = 1$. Then for all primes p,

$$\zeta_{L_p}^{\mathrm{iso}}(s) = \int_{G_p^+} |\det g|_p^s \, \mathrm{d}\mu_{G_p}(g) \tag{3.2}$$

where $\zeta_{L_p}^{\mathrm{iso}}(s)$ enumerates Lie sublattices that are isomorphic to L_p.

We may decompose the 1-component $\mathbf{G}^\circ$ into a semidirect product $\mathbf{G}^\circ = \mathbf{N} \rtimes \mathbf{H}$ of its unipotent radical $\mathbf{N}$ and a reductive group $\mathbf{H}$; compare with [21, §VIII.4]. Fix a prime p and write $G = \mathbf{G}(\mathbb{Q}_p)$, $N = \mathbf{N}(\mathbb{Q}_p)$, $H = \mathbf{H}(\mathbb{Q}_p)$. Let $V = {}_{\mathbb{Q}_p}L \cong \mathbb{Q}_p^d$ be the $\mathbb{Q}_p^d$-vector space on which G acts from the right. In [14, §2], du Sautoy and Lubotzky provide a general framework for reducing an integral of the form (3.2) to an integral over a suitable subset $H^+ \subseteq H$. Their reduction depends, in general, on several technical assumptions (some of which can be realised by excluding finitely many primes):

(a) $\mathbf{G} = \mathbf{G}^\circ$ is connected.

(b) There exists a vector space decomposition $V = \bigoplus_{i=1}^c U_i$, with associated flag $V_j = \bigoplus_{i=j}^c U_i$, $1 \leq j \leq c+1$, such that each U_i is H-invariant, each V_j is N-invariant and the induced action of N on each quotient V_j/V_{j+1}, $1 \leq j \leq c$, is trivial.

(c) A certain lifting condition holds with respect to this decomposition; see [14, Assumption 2.3] for a complete description and Condition 3.4 below for a specific instance.

The requirement that the action of N on the quotients V_j/V_{j+1} be trivial is not actually needed for the reduction. However, it is usually desirable—for both theoretical and practical applications. We will shortly see that in our applications we need to drop this requirement.

We now specialise to the case where L is a D^*-Lie lattice associated, via (2.2) above, to the polynomial $\Delta(t) = t^m$ for some integer $m \geq 2$. Note that L is a class-two nilpotent $\mathbb{Z}$-Lie lattice of rank $d = 2m+2$ with rank-two centre and $\mathrm{Z}(L) = [L, L]$. Our aim is to identify modified versions of the above technical assumptions in order to carry out a reduction of the integral in the spirit of

du Sautoy and Lubotzky, without excluding any primes. In our setting, $\mathbf{G}$ is connected and the splitting $\mathbf{G} = \mathbf{N} \rtimes \mathbf{H}$ is very explicit; see Corollary 2.7. Thus we are not worried about (a). We write $V = U_1 \oplus U_2$, where $U_2 = [{}_{\mathbb{Q}_p}L, {}_{\mathbb{Q}_p}L]$ and U_1 is an H-stable complement to U_2 in V, corresponding to the abelianisation of ${}_{\mathbb{Q}_p}L$; in the case of interest to us, U_1 is the $\mathbb{Q}_p$-span of a natural set of generators for the Lie lattice L_p. Note that U_2 is automatically invariant under the action of G, while U_1 is H-invariant by construction; however, our decomposition is 'coarse' in the sense that the actions of N on V/U_2 and on U_2 are not trivial as stipulated in (b).

We now go about describing a weak version of (c) that suffices for our purposes. Remarkably, [14, Assumption 2.3] does not apply to the D^*-Lie lattice associated to t^3; compare with Remark 5.6 below. Let $N_1 = N \cap \ker(\psi_2')$, where $\psi_2' \colon G \to \mathrm{Aut}(V/U_2)$ denotes the natural action. Since U_2 is N-invariant, we may define the induced map $\psi_2 \colon G/N_1 \to \mathrm{Aut}(V/U_2) \leq \mathsf{GL}_{2m}(\mathbb{Q}_p)$, and the set

$$(G/N_1)^+ = \psi_2^{-1}(\psi_2(G/N_1) \cap \mathsf{M}_{2m}(\mathbb{Z}_p)),$$

where $2m = \dim V/U_2$ is the dimension of the abelianisation of L_p.

CONDITION 3.4: *For every $g_0 N_1 \in (G/N_1)^+$ there exists $g \in G^+$ such that $g_0 N_1 = g N_1$.*

Remark 3.5: The effect of Condition 3.4 is weaker than that of [14, Assumption 2.3], because in our situation N does not act trivially on V/U_2. Condition 3.4 is trivially satisfied due to the freedom to replace g_0 by $g \in g_0 N_1$ such that vg has zero component in U_2 for all $v \in U_1$. In matrix terms, this amounts to replacing the top-right block 'above the centre' by zeros. The action of g_0 and g on U_2 is the same and induced by the action on V/U_2; as the action on V/U_2 is 'integral', it is also integral on U_2.

Define $\vartheta_0 \colon H \to \mathbb{R}_{\geq 0}$ by setting

$$\vartheta_0(h) = \mu_{N/N_1}(\{uN_1 \in N/N_1 \mid uhN_1 \in (G/N_1)^+\}),$$

where μ_{N/N_1} denotes the right Haar measure on N/N_1, normalised such that the set $\psi_2^{-1}(\psi_2(N/N_1) \cap \mathsf{M}_{2m}(\mathbb{Z}_p))$ has measure 1. Similarly, define $\vartheta_1 \colon H \to \mathbb{R}_{\geq 0}$ by setting

$$\vartheta_1(h) = \mu_{N_1}(\{u \in N_1 \mid nh \in G^+\}),$$

where μ_{N_1} denotes the right Haar measure on N_1, normalised such that the set $N_1^+ = N_1(\mathbb{Z}_p)$ has measure 1.

Write μ_G, respectively μ_H, for the right Haar measure on G, respectively H, normalised such that $\mu_G(\mathbf{G}(\mathbb{Z}_p)) = 1$, respectively $\mu_H(\mathbf{H}(\mathbb{Z}_p)) = 1$. From $G = N \rtimes H$ one deduces (using Condition 3.4 and Remark 3.5) that

$$\mu_G = \mu_{N/N_1} \cdot \mu_{N_1} \cdot \mu_H.$$

Setting $G^+ = G \cap \mathsf{M}_{2m+2}(\mathbb{Z}_p)$ and $H^+ = H \cap \mathsf{M}_{2m+2}(\mathbb{Z}_p)$, one obtains the following by a mild adaptation of the proof of [14, Thm. 2.2] to the coarse decomposition $V = U_1 \oplus U_2$.

THEOREM 3.6: *In the set-up described above, we have*

$$\int_{G^+} |\det g|_p^s \, \mathrm{d}\mu_G(g) = \int_{H^+} |\det h|_p^s \, \vartheta_0(h) \, \vartheta_1(h) \, \mathrm{d}\mu_H(h).$$

In our applications we will see that $\vartheta_1(h)$ is straightforward to calculate, while $\vartheta_0(h)$ appears to be rather complicated to track down for large m. For short, we set $\vartheta(h) = \vartheta_0(h)\vartheta_1(h)$ for $h \in H$. In view of [14, Thm. 2.3], one could suspect the function $\vartheta\colon H \to \mathbb{R}_{>0}$ to be a character on H, but it was demonstrated in [6] that, for general class-two nilpotent groups, one cannot expect this to be the case. Indeed, in Sections 4 and 5 we will see that ϑ is a character for the group Γ_{t^2}, but that it is not a character for the group Γ_{t^3}; see Remark 5.6. Subject to the modifications detailed above, the three technical assumptions (a), (b), (c) of [14, §2] are, indeed, satisfied in our setting for every prime p. For a general class-two nilpotent Lie lattice, our methods leading to Theorem 3.6 work for almost all primes p and may prove to be useful in other contexts, where [14, Assumption 2.3] does not hold.

3.3. UTILISING A p-ADIC BRUHAT DECOMPOSITION. We recall the machinery developed by Igusa [23], du Sautoy and Lubotzky [14] and the first author [4] for utilising a p-adic Bruhat decomposition in order to compute integrals over reductive groups; the reference [4] is useful for practical purposes, where the notation (and some further choices) are well-suited to the current paper. We apply this theory in Sections 4 and 5.

Suppose that the group $\mathbf{H}$ is isomorphic to an affine $\mathbb{Z}$-group scheme $\dot{\mathbf{H}} \leq \mathsf{GL}_{\dot{d}}$ and denote by $\varrho\colon \dot{\mathbf{H}} \to \mathbf{H}$ a corresponding isomorphism. In our applications, we have $\dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1 \leq \mathsf{GL}_3$ and ϱ is the isomorphism described in Corollary 2.7. It is useful to keep this special situation in mind for a concrete interpretation of the following general approach. We write $\dot{H} = \dot{\mathbf{H}}(\mathbb{Q}_p)$, equipped with the

right Haar measure $\mu_{\dot{H}}$ normalised such that $\mu_{\dot{H}}(\dot{\mathbf{H}}(\mathbb{Z}_p)) = 1$. We take interest in the p-adic integral

$$\mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,p}(s) = \int_{H^+\varrho^{-1}} |\det h^{\varrho}|_p^s \, \vartheta(h^{\varrho}) \, \mathrm{d}\mu_{\dot{H}}(h),$$

where $H^+\varrho^{-1}$ denotes the full pre-image of H^+ under ϱ (in the literature this pre-image is usually denoted by $\dot{H}^+$, for short, but we prefer the more descriptive form to avoid misunderstandings). In our applications, ϱ induces a measure-preserving map from $\dot{H}$ to H, as $\dot{\mathbf{H}}(\mathbb{Z}_p)\varrho = \mathbf{H}(\mathbb{Z}_p)$; in this situation, one could even get away with 'identifying' $\mathbf{H}$ and $\dot{\mathbf{H}}$.

We fix a maximal torus $\mathbf{T}$ in $\dot{\mathbf{H}}$ and assume that $\mathbf{T}$ splits over $\mathbb{Q}$; this can be arranged in our applications. Under an assumption of 'good reduction', elements of $\mathbf{T}$ act by conjugation on minimal closed unipotent subgroups of $\dot{\mathbf{H}}$; this action gives rise to a root system $\Phi \subseteq \mathrm{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}})$. The (finite) Weyl group W of $\dot{\mathbf{H}}$ corresponds to $\mathrm{N}_{\dot{\mathbf{H}}}(\mathbf{T})/\mathbf{T}$, where $\mathrm{N}_{\dot{\mathbf{H}}}(\mathbf{T})$ is the normaliser of $\mathbf{T}$ in $\dot{\mathbf{H}}$. We suppress here some necessary requirements of good reduction since these will all trivially hold in our applications; the technical requirements are detailed in [4]. We choose a set of simple roots $\alpha_1, \ldots, \alpha_\ell$ which define the positive roots Φ^+. Let $\Xi = \mathrm{Hom}(\mathsf{G}_{\mathrm{m}}, \mathbf{T})$ denote the set of co-characters of $\mathbf{T}$. We refer to [14] for a description of the Iwahori subgroup $\mathcal{B} \leq \dot{\mathbf{H}}(\mathbb{Z}_p)$ with respect to the simple roots $\alpha_1, \ldots, \alpha_\ell$. Let π denote a fixed uniformising parameter for $\mathbb{Z}_p$, e.g., $\pi = p$. The p-adic Bruhat decomposition theorem of Iwahori and Matsumoto [24] gives

$$\dot{H} = \dot{\mathbf{H}}(\mathbb{Q}_p) = \coprod_{\substack{w \in W \\ \xi \in \Xi}} \mathcal{B}\, w\, \xi(\pi)\, \mathcal{B} \quad \text{and} \quad \dot{\mathbf{H}}(\mathbb{Z}_p) = \coprod_{w \in W} \mathcal{B}\, w\, \mathcal{B},$$

where elements $w \in W$ in this context are to be read as coset representatives $g_w \in \mathrm{N}_{\dot{\mathbf{H}}}(\mathbf{T})(\mathbb{Z}_p)$. One defines $\Xi^+ = \{\xi \in \Xi \mid \xi(\pi) \in H^+\varrho^{-1}\}$ and considers, for $w \in W$,

$$w\Xi_w^+ = \{\xi \in \Xi^+ \mid \alpha_i(\xi(\pi)) \in \mathbb{Z}_p \text{ for } 1 \leq i \leq \ell, \\ \text{and } \alpha_i(\xi(\pi)) \in p\mathbb{Z}_p \text{ whenever } \alpha_i \in w(\Phi^-)\},$$

where Φ^- denotes the set of negative roots. Utilising symmetries in the affine Weyl group and the fact that $|\det \cdot^{\varrho}|_p$, $\vartheta(\cdot^{\varrho})$ are constant on double cosets of $\mathcal{B} \leq \dot{\mathbf{H}}(\mathbb{Z}_p)$, (compare with [4, Lem. 3.10]) the following generalisation of [14, (5.4)] holds.

PROPOSITION 3.7 (du Sautoy, Lubotzky; Berman [4, Prop. 4.2]): *If* $\mathbf{T}$ *splits over* $\mathbb{Q}$ *then, assuming good reduction,*

$$\mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,p}(s) = \sum_{w\in W} p^{-\operatorname{len}(w)} \sum_{\xi\in w\Xi^+_w} \left| \Big(\prod_{\beta\in\Phi^+} \beta \Big)(\xi(\pi)) \right|_p^{-1} |\det \xi(\pi)^{\varrho}|_p^s \, \vartheta(\xi(\pi)^{\varrho}),$$

where $\operatorname{len}(\cdot)$ *is the standard length function on* W*.*

Finally we recall a natural pairing between $\Xi = \operatorname{Hom}(\mathsf{G}_{\mathrm{m}}, \mathbf{T})$ and $\operatorname{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}})$: this is the map $(\beta, \xi) \mapsto \langle \beta, \xi \rangle$, where $\beta(\xi(\tau)) = \tau^{\langle \beta, \xi \rangle}$ for all $\tau \in \mathsf{G}_{\mathrm{m}}$. As in [4, §5.2], it will turn out to be convenient to judiciously choose a basis for $\operatorname{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}})$, consisting of simple roots and dominant weights for the contragredient representations of irreducible components of ϱ, and then to determine a dual basis for $\operatorname{Hom}(\mathsf{G}_{\mathrm{m}}, \mathbf{T})$. This will enable an explicit description of the set $w\Xi^+_w$.

Example 3.8: To illustrate the general set-up, we indicate how it can be used to compute the pro-isomorphic zeta function of the D^*-group $\Gamma = \Gamma_t$ of Hirsch length 4, defined in (1.3). Proposition 3.1 shows that $\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s)$ for all primes p; here L is the $\mathbb{Z}$-Lie lattice of $\mathbb{Z}$-rank 4, defined by (2.2) with respect to the $\mathbb{Z}$-basis $\mathcal{S}$, where $K = (0)$ is the companion matrix of the prime polynomial $\Delta_K = t$. We consider the algebraic automorphism group $\mathbf{G} = \mathbf{Aut}(L)$, with respect to the $\mathbb{Z}$-basis $\mathcal{S}^* = (x_1, y_1, z_2, z_1)$ as in Corollary 2.7 and Example 2.9.

Let p be a prime; our aim is to calculate the local pro-isomorphic zeta function $\zeta^{\wedge}_{L,p}(s)$. The coarse decomposition of $V = {}_{\mathbb{Q}_p}L$ described in Section 3.2 is not suitable, due to the fact that here the centre does not coincide with the derived sublattice of L. Instead we require a refined decomposition. Setting

$$U_1 = \operatorname{span}_{\mathbb{Q}_p}\{x_1, y_1\}, \quad U_2 = \operatorname{span}_{\mathbb{Q}_p}\{z_2\}, \quad U_3 = \operatorname{span}_{\mathbb{Q}_p}\{z_1\},$$

we write

$$G = \mathbf{G}(\mathbb{Q}_p), \quad H = \mathbf{H}(\mathbb{Q}_p), \quad N = \mathbf{N}(\mathbb{Q}_p);$$

these groups act on $V = {}_{\mathbb{Q}_p}L = U_1 \oplus U_2 \oplus U_3$ in a suitable way. We now require the following subgroups of the unipotent radical: $N_1 = N \cap \ker(\psi'_2)$, where $\psi'_2 \colon G \to \operatorname{Aut}(V/(U_2 + U_3))$ denotes the natural action, and $N_2 = N \cap \ker(\psi'_3)$, where $\psi'_3 \colon G \to \operatorname{Aut}(V/U_3)$ denotes the natural action. By Corollary 2.7, the elements of the reductive subgroup H are of the form

$$\operatorname{diag}(A, \nu \det A, \det A), \quad \text{where } (A, \nu) \in \mathsf{GL}_2(\mathbb{Q}_p) \times \mathsf{GL}_1(\mathbb{Q}_p), \tag{3.3}$$

and, according to Example 2.9, elements of N take the form

$$\left(\begin{array}{c|cc} \mathrm{I}_2 & * & * \\ \hline 0 & 1 & \lambda \\ 0 & 0 & 1 \end{array}\right), \quad \text{with } \lambda \in \mathbb{Q}_p \text{ and arbitrary entries in the positions marked } *.$$

As explained above, we can utilise Proposition 3.3 and Theorem 3.6 to compute $\zeta^{\wedge}_{L,p}(s)$ via an integral over H^+. A short calculation (using a slightly different analysis of ϑ, based on [14, §2] with respect to the decomposition $U_1 \oplus U_2 \oplus U_3$) shows that, for $h \in H^+$ of the form (3.3),

$$\vartheta(h) = |\det A|_p^{-5} |\nu|_p^{-2}.$$

From here on a direct calculation could be carried out; but we prefer to illustrate the use of the Bruhat decomposition. We observe that the morphism

$$\varrho \colon \dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1 \to \mathbf{H}, \quad (A, \nu) \mapsto \mathrm{diag}(A, \nu \det A, \det A)$$

induces a measure-preserving isomorphism $\dot{H} = \dot{\mathbf{H}}(\mathbb{Q}_p) \to H$ such that

$$H^+ \varrho^{-1} = \{(A, \nu) \mid v_p(A) \geq 0 \text{ and } v_p(\det A) + v_p(\nu) \geq 0\},$$

where $v_p \colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ denotes in the first place the standard p-adic valuation map and also the map $\mathsf{M}_2(\mathbb{Q}_p) \to \mathbb{Z} \cup \{\infty\}$, $(a_{ij}) \mapsto \min\{v_p(a_{ij}) \mid 1 \leq i, j \leq 2\}$. Thus we obtain

$$\zeta^{\wedge}_{L,p}(s) = \int_{\substack{(A,\nu) \in \dot{H} \\ \text{with } v_p(A) \geq 0, \\ v_p(\det A) + v_p(\nu) \geq 0}} |\det A|_p^{3s-5} \, |\nu|_p^{s-2} \, \mathrm{d}\mu_p(A, \nu).$$

For convenience, we consider $\dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1$ as a subgroup of GL_3, embedded as block matrices via $(A, \nu) \mapsto \mathrm{diag}(A, \nu)$. In particular,

$$T = \mathbf{T}(\mathbb{Q}_p) = \{\mathrm{diag}(\lambda_1, \lambda_2, \nu) \mid \lambda_1, \lambda_2, \nu \in \mathbb{Q}_p^{\times}\}$$

is a maximal torus in $\dot{H}$. By Proposition 3.7 we obtain

$$\zeta^{\wedge}_{L,p}(s) = \sum_{w \in W} p^{-\operatorname{len}(w)} \sum_{\xi \in w\Xi^+_w} |\alpha(\xi(\pi))|_p^{-1} \, |\det(\xi(\pi)^{\varrho})|_p^{s} \, \vartheta(\xi(\pi)^{\varrho}),$$

where we choose $\alpha \in \mathrm{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}})$, $\alpha(\mathrm{diag}(\lambda_1, \lambda_2, \nu)) = \lambda_1 \lambda_2^{-1}$ as the single positive root, and we have

$$w\Xi^+_w = \{\xi \in \Xi^+ \mid \alpha(\xi(\pi)) \in \mathbb{Z}_p, \text{ and } \alpha(\xi(\pi)) \in p\mathbb{Z}_p \text{ if } w = w_0\},$$

where the Weyl group is $W = \{1, w_0\}$. In order to describe the set $w\Xi_w^+$ we consider dominant weights for the contragredient representation, following [14]. These are given by

$$\omega_1^{-1}(h) = \lambda_2, \quad \omega_2^{-1}(h) = \lambda_1\lambda_2\nu \quad \text{for } h = \operatorname{diag}(\lambda_1, \lambda_2, \nu) \in T.$$

It follows that $\alpha, \omega_1^{-1}, \omega_2^{-1}$ form a $\mathbb{Z}$-basis for $\operatorname{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}})$ whose $\mathbb{N}_0$-span contains all the weights of ϱ. Thus to detect whether an element $h \in T$ is integral it is sufficient to check whether $\alpha(h), \omega_1^{-1}(h), \omega_2^{-1}(h)$ all lie in $\mathbb{Z}_p$. We rewrite $\alpha_1 = \alpha$, $\alpha_2 = \omega_1^{-1}$, $\alpha_3 = \omega_2^{-1}$ and find that $\xi_1, \xi_2, \xi_3 \in \Xi$ defined by

$$\xi_1(\tau) = (\tau, 1, \tau^{-1}), \quad \xi_2(\tau) = (\tau, \tau, \tau^{-2}), \quad \xi_3(\tau) = (1, 1, \tau) \quad \text{for } \tau \in \mathbb{Q}_p^\times$$

form a dual basis so that

$$\langle \alpha_i, \xi_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

A general element of Ξ has the form $\xi_{\mathbf{e}} = \xi_1^{e_1}\xi_2^{e_2}\xi_3^{e_3}$ with $\mathbf{e} = (e_1, e_2, e_3) \in \mathbb{Z}^3$ and satisfies $\xi_{\mathbf{e}}(\pi) = \operatorname{diag}(\pi^{e_1+e_2}, \pi^{e_2}, \pi^{-e_1-2e_2+e_3})$. Hence

$$\xi_{\mathbf{e}}(\pi)^\varrho = \operatorname{diag}(\pi^{e_1+e_2}, \pi^{e_2}, \pi^{e_3}, \pi^{e_1+2e_2})$$

and we read off

$$|\det \xi_{\mathbf{e}}(\pi)^\varrho|_p^s = p^{-(2e_1+4e_2+e_3)s}, \quad \vartheta(\xi_{\mathbf{e}}(\pi)^\varrho) = p^{3e_1+6e_2+2e_3}.$$

Note that $|\alpha(\xi_{\mathbf{e}}(\pi))|^{-1} = p^{\langle \alpha, \xi_{\mathbf{e}} \rangle} = p^{\langle \alpha_1, \xi_{\mathbf{e}} \rangle} = p^{e_1}$ and we can rewrite

$$w\Xi_w^+ = \{\xi \in \Xi \mid \langle \alpha_i, \xi \rangle \geq 0 \text{ for } i \in \{1, 2, 3\}, \text{ and } \langle \alpha_1, \xi \rangle > 0 \text{ if } w = w_0\},$$

since $\alpha_1 \in w(\Phi^-)$ if and only if $w \neq 1$. Thus we obtain

$$\begin{aligned}
\mathcal{Z}_{\dot{\mathbf{H}}, \varrho, \vartheta, p}(s) &= \sum_{w \in W} p^{-\operatorname{len}(w)} \sum_{\xi \in w\Xi_w^+} p^{\langle \alpha, \xi \rangle} \, |\det \xi(\pi)^\varrho|_p^s \, \vartheta(\xi(\pi)^\varrho) \\
&= \sum_{w \in W} p^{-\operatorname{len}(w)} \sum_{\substack{\mathbf{e} \in \mathbb{N}_0^3 \text{ with} \\ e_1 > 0 \text{ if } w \neq 1}} p^{(4-2s)e_1 + (6-4s)e_2 + (2-s)e_3} \\
&= \frac{1}{(1-p^{6-4s})(1-p^{2-s})} \Big(p^0 \cdot \frac{1}{1-p^{4-2s}} + p^{-1} \cdot \frac{p^{4-2s}}{1-p^{4-2s}} \Big) \\
&= \frac{1}{(1-p^{3-2s})(1-p^{4-2s})(1-p^{2-s})},
\end{aligned}$$

confirming the formula that we reported in the introduction, based on [3, §3.3.4].

4. The local pro-isomorphic zeta functions of the group Γ_{t^2}

In this section we consider the pro-isomorphic zeta function of the D^*-group $\Gamma = \Gamma_{t^2}$ of Hirsch length 6, defined in (1.3). We prove Theorem 1.1 and obtain Corollary 1.2; it turns out that we can proceed as in Example 3.8, taking care of a little extra complexity along the way.

Proposition 3.1 shows that $\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s)$ for all primes p, where L is the $\mathbb{Z}$-Lie lattice associated to Γ. In our setting, L is the $\mathbb{Q}$-indecomposable D^*-Lie lattice L of $\mathbb{Z}$-rank 6, defined by (2.2) with respect to the $\mathbb{Z}$-basis $\mathcal{S}$, where $K = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the companion matrix of the primary polynomial $\Delta_K = t^2$. We consider the algebraic automorphism group $\mathbf{G} = \mathbf{Aut}(L)$, with respect to the $\mathbb{Z}$-basis $\mathcal{S}^* = (x_1, y_2, x_2, y_1, z_2, z_1)$ as in Corollary 2.7 and Example 2.9.

Let p be a prime; we will set about calculating the local pro-isomorphic zeta function $\zeta^{\wedge}_{L,p}(s)$. In the notation of Section 3, we set $U_1 = \mathrm{span}_{\mathbb{Q}_p}\{x_1, y_2, x_2, y_1\}$ and $U_2 = \mathrm{span}_{\mathbb{Q}_p}\{z_2, z_1\}$. We write

$$G = \mathbf{G}(\mathbb{Q}_p), \quad H = \mathbf{H}(\mathbb{Q}_p), \quad N = \mathbf{N}(\mathbb{Q}_p);$$

these groups act on $V = {}_{\mathbb{Q}_p}L = U_1 \oplus U_2$. By Corollary 2.7, the elements of the reductive subgroup H are of the form

$$(4.1) \qquad \left(\begin{array}{cc|cc} \nu A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ \hline 0 & 0 & \nu^2 \det A & 0 \\ 0 & 0 & 0 & \nu \det A \end{array}\right), \quad \text{where } (A, \nu) \in \mathsf{GL}_2(\mathbb{Q}_p) \times \mathsf{GL}_1(\mathbb{Q}_p).$$

The description of the unipotent radical given in Example 2.9 shows that elements of N are of the form

$$\left(\begin{array}{cc|cc} \mathrm{I}_2 & B & * & * \\ 0 & \mathrm{I}_2 & * & * \\ \hline 0 & 0 & 1 & \mathrm{tr}\, B \\ 0 & 0 & 0 & 1 \end{array}\right), \quad \text{where } B \in \mathsf{M}_2(\mathbb{Q}_p)$$

and there are arbitrary entries in the positions marked $*$. As explained in Section 3, we can utilise Proposition 3.3 and Theorem 3.6 to compute $\zeta^{\wedge}_{L,p}(s)$ via an integral over H^+.

We now set about calculating the functions ϑ_0, ϑ_1 defined in Section 3; we refer to Section 3.2 for definitions of N_1, μ_{N/N_1} and μ_{N_1}. Noting that

$$N/N_1 \cong \mathbb{Q}_p^4 \quad \text{and} \quad N_1 \cong \mathbb{Q}_p^8,$$

we obtain for $h \in H^+$ of the form (4.1)

$$\vartheta_0(h) = |\det A|_p^{-2} \quad \text{and} \quad \vartheta_1(h) = |\nu^3 \det A^2|_p^{-4},$$

hence

$$\vartheta(h) = \vartheta_0(h)\vartheta_1(h) = |\det A|_p^{-10}|\nu|_p^{-12};$$

in particular, $\vartheta\colon H \to \mathbb{R}_{>0}$ is a character.

We observe that the morphism

$$\varrho\colon \dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1 \to \mathbf{H}, \quad (A, \nu) \mapsto \operatorname{diag}(\nu A, A, \nu^2 \det A, \nu \det A)$$

induces a measure-preserving isomorphism $\dot{H} = \dot{\mathbf{H}}(\mathbb{Q}_p) \to H$ such that

$$H^+\varrho^{-1} = \{(A, \nu) \mid v_p(A) \geq 0 \text{ and } v_p(A) + v_p(\nu) \geq 0\},$$

where (as in Example 3.8) $v_p\colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ denotes the standard p-adic valuation map as well as the map $\mathsf{M}_2(\mathbb{Q}_p) \to \mathbb{Z} \cup \{\infty\}$,

$$(a_{ij}) \mapsto \min\{v_p(a_{ij}) \mid 1 \leq i, j \leq 2\}.$$

Thus we obtain

$$\zeta_{L,p}^{\wedge}(s) = \int\limits_{\substack{(A,\nu)\in\dot{H} \\ \text{with } v_p(A)\geq 0, \\ v_p(A)+v_p(\nu)\geq 0}} |\det A|_p^{4s-10}\, |\nu|_p^{5s-12}\, \mathrm{d}\mu_p(A, \nu). \tag{4.2}$$

For convenience, we consider $\dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1$ as a subgroup of GL_3, embedded as block matrices via $(A, \nu) \mapsto \operatorname{diag}(A, \nu)$. In particular,

$$T = \mathbf{T}(\mathbb{Q}_p) = \{\operatorname{diag}(\lambda_1, \lambda_2, \nu) \mid \lambda_1, \lambda_2, \nu \in \mathbb{Q}_p^\times\}$$

is a maximal torus in $\dot{H}$.

By Proposition 3.7 we obtain

$$\zeta_{L,p}^{\wedge}(s) = \sum_{w\in W} p^{-\operatorname{len}(w)} \sum_{\xi\in w\Xi_w^+} |\alpha(\xi(\pi))|_p^{-1}\ |\det(\xi(\pi)^\varrho)|_p^s\ \vartheta(\xi(\pi)^\varrho),$$

where we choose $\alpha \in \operatorname{Hom}(\mathbf{T}, \mathsf{G}_\mathrm{m}), \alpha(\operatorname{diag}(\lambda_1, \lambda_2, \nu)) = \lambda_1\lambda_2^{-1}$ as the single positive root, and we have

$$w\Xi_w^+ = \{\xi \in \Xi^+ \mid \alpha(\xi(\pi)) \in \mathbb{Z}_p, \text{ and } \alpha(\xi(\pi)) \in p\mathbb{Z}_p \text{ if } w = w_0\},$$

where the Weyl group is $W = \{1, w_0\}$. In order to describe the set $w\Xi_w^+$ we will need to consider dominant weights for the contragredient representation, following [14]. These are given by

$$\omega_1^{-1}(h) = \lambda_2\nu, \quad \omega_2^{-1}(h) = \lambda_2, \quad \omega_3^{-1}(h) = \lambda_1\lambda_2\nu^2, \quad \omega_4^{-1}(h) = \lambda_1\lambda_2\nu$$

for $h = \mathrm{diag}(\lambda_1, \lambda_2, \nu) \in T$. It follows that $\alpha, \omega_1^{-1}, \omega_2^{-1}$ form a $\mathbb{Z}$-basis for $\mathrm{Hom}(\mathbf{T}, \mathsf{G}_\mathrm{m})$ whose $\mathbb{N}_0$-span contains all the weights of ϱ. Thus to detect whether an element $h \in T$ is integral it is sufficient to check whether $\alpha(h), \omega_1^{-1}(h), \omega_2^{-1}(h)$ all lie in $\mathbb{Z}_p$. We rewrite $\alpha_1 = \alpha$, $\alpha_2 = \omega_1^{-1}$, $\alpha_3 = \omega_2^{-1}$ and seek a dual basis, namely elements $\xi_1, \xi_2, \xi_3 \in \Xi$ such that

$$\langle \alpha_i, \xi_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

A routine calculation shows that the following elements suffice:

$$\xi_1(\tau) = (\tau, 1, 1), \quad \xi_2(\tau) = (1, 1, \tau), \quad \xi_3(\tau) = (\tau, \tau, \tau^{-1}) \quad \text{for } \tau \in \mathbb{Q}_p^\times.$$

A general element of Ξ has the form $\xi_\mathbf{e} = \xi_1^{e_1}\xi_2^{e_2}\xi_3^{e_3}$ with $\mathbf{e} = (e_1, e_2, e_3) \in \mathbb{Z}^3$ and satisfies $\xi_\mathbf{e}(\pi) = \mathrm{diag}(\pi^{e_1+e_3}, \pi^{e_3}, \pi^{e_2-e_3})$. Hence

$$\xi_\mathbf{e}(\pi)^\varrho = \mathrm{diag}(\pi^{e_1+e_2}, \pi^{e_2}, \pi^{e_1+e_3}, \pi^{e_3}, \pi^{e_1+2e_2}, \pi^{e_1+e_2+e_3})$$

and we read off

$$|\det \xi_\mathbf{e}(\pi)^\varrho|_p^s = p^{-(4e_1+5e_2+3e_3)s}, \quad \vartheta(\xi_\mathbf{e}(\pi)^\varrho) = p^{10e_1+12e_2+8e_3}.$$

Note that $|\alpha(\xi_\mathbf{e}(\pi))|^{-1} = p^{\langle \alpha, \xi_\mathbf{e} \rangle} = p^{\langle \alpha_1, \xi_\mathbf{e} \rangle} = p^{e_1}$ and we can rewrite

$$w\Xi_w^+ = \{\xi \in \Xi \mid \langle \alpha_i, \xi \rangle \geq 0 \text{ for } i \in \{1, 2, 3\}, \text{ and } \langle \alpha_1, \xi \rangle > 0 \text{ if } w = w_0\},$$

since $\alpha_1 \in w(\Phi^-)$ if and only if $w \neq 1$. Thus we obtain

$$\begin{aligned}
\mathcal{Z}_{\dot{\mathbf{H}}, \varrho, \vartheta, p}(s) &= \sum_{w \in W} p^{-\operatorname{len}(w)} \sum_{\xi \in w\Xi_w^+} p^{\langle \alpha, \xi \rangle} |\det \xi(\pi)^\varrho|_p^s \, \vartheta(\xi(\pi)^\varrho) \\
&= \sum_{w \in W} p^{-\operatorname{len}(w)} \sum_{\substack{\mathbf{e} \in \mathbb{N}_0^3 \\ \text{with } e_1 > 0 \text{ if } w \neq 1}} p^{(11-4s)e_1 + (12-5s)e_2 + (8-3s)e_3} \\
&= \frac{1}{(1-p^{12-5s})(1-p^{8-3s})} \Big(p^0 \cdot \frac{1}{1-p^{11-4s}} + p^{-1} \cdot \frac{p^{11-4s}}{1-p^{11-4s}} \Big) \\
&= \frac{1+p^{10-4s}}{(1-p^{8-3s})(1-p^{11-4s})(1-p^{12-5s})},
\end{aligned}$$

proving Theorem 1.1. The first part of Corollary 1.2 follows directly from well-known properties of the Riemann zeta function. For the assertion about the asymptotic growth of pro-isomorphic subgroups in Γ, we use a Tauberian theorem as recorded in [12, Thm. 4.20]. In the notation employed there, we take $a = 3$, $g(s) = (\frac{1}{12} + g_1(s))\zeta(5s-12)\zeta(4s-10)/\zeta(8s-20)$ with $g_1(s)$ holomorphic such that $g_1(3) = 0$, and $w = 2$ so that (1.4) holds for

$$c_{t^2} = \frac{g(a)}{a\Gamma(w)} = \frac{\frac{1}{12}}{3\Gamma(2)}\frac{\zeta(3)\zeta(2)}{\zeta(4)};$$

using the precise values $\Gamma(2) = 1$, $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ and the estimate $\zeta(3) \approx 1.202057$ we arrive at the claimed description of the invariant c_{t^2}.

5. The local pro-isomorphic zeta functions of the group Γ_{t^3}

5.1. Counting points on a quadratic surface. In preparation for computing the pro-isomorphic zeta function of the group Γ_{t^3}, we study a certain arithmetic function. In order to make the analysis transferable to a more general setting, considered in Section 7, we work over a compact discrete valuation ring $\mathfrak{o}$ with maximal ideal $\wp = \pi\mathfrak{o}$ and residue field $\mathfrak{o}/\wp \cong \mathbb{F}_q$ of size q and characteristic p. Our primary interest is in the basic set-up:

$$\mathfrak{o} = \mathbb{Z}_p, \quad \wp = p\mathbb{Z}_p, \quad \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p.$$

Definition 5.1: For $\alpha, \beta, m \in \mathbb{N}_0$ and an indeterminate t, let

$$f(\alpha,\beta,m) = \#\{(x,y,z) \in (\mathfrak{o}/\pi^m\mathfrak{o})^3 \mid \pi^\alpha x^2 + \pi^\beta yz = 0\},$$

$$F_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} f(\alpha,\beta,m)t^m,$$

and, for $\beta \in \mathbb{N}_0$, let

$$F^\star_{0,\beta}(t) = \sum_{m=\beta}^{\infty} f(0,\beta,m)t^m.$$

Observing that for $\alpha, \beta, m \in \mathbb{N}_0$ one trivially has

$$f(\alpha+1,\beta+1,m+1) = q^3 f(\alpha,\beta,m), \tag{5.1}$$

we focus on the cases where either α or β is zero.

PROPOSITION 5.2: *For $\alpha \in \mathbb{N}_0$, we have*

(i) $F_{\alpha,0}(t) = q^{2\alpha}t^{\alpha}F_{0,0}(t) + \frac{(1-qt)(1-q^{2\alpha}t^{\alpha})}{(1-q^2t)^2}$,

(ii) $F_{0,\alpha}(t) = (q^5t^2)^{\lfloor \alpha/2 \rfloor}F_{0,\overline{\alpha}}(t) + (1+q^2t)\frac{1-q^{5\lfloor \frac{\alpha}{2} \rfloor}t^{2\lfloor \frac{\alpha}{2} \rfloor}}{1-q^5t^2}$,

where $\overline{\alpha} = 0$ for α even and $\overline{\alpha} = 1$ for α odd. In particular,

$$F_{0,0}(t) = \frac{1-q^2t^2}{(1-q^2t)(1-q^3t^2)} \quad \text{and} \quad F_{0,1}(t) = \frac{1-2q^3t^2+q^4t^2}{(1-q^2t)(1-q^3t^2)}.$$

To prove Proposition 5.2 we use the following recurrence relations. Parts (1) and (2) of Lemma 5.3 below form the basis for the recursion in α, β given in (3) and (4). Together with (5.1) they determine $f(\alpha, \beta, m)$ completely.

LEMMA 5.3: *For $\alpha, \beta, m \in \mathbb{N}_0$ the following hold:*

(1) $f(0,0,m+2) = q^2(q^2-1)q^{2m} + q^3f(0,0,m)$,
(2) $f(0,1,m+2) = 2q^3(q-1)q^{2m} + q^3f(0,1,m)$,
(3) $f(0,\beta+2,m+2) = q^5f(0,\beta,m)$,
(4) $f(\alpha+1,0,m+1) = q(q-1)q^{2m} + q^2f(\alpha,0,m)$.

Proof. To prove (1), we observe that for the finite field $\mathbb{F}_q$, the set of $\mathbb{F}_q$-rational points of the affine variety defined by $x^2 + yz$, viz.

$$\{(x, y, -y^{-1}x^2) \mid x \in \mathbb{F}_q, y \in \mathbb{F}_q^{\times}\} \cup \{(0,0,z) \mid z \in \mathbb{F}_q\},$$

has q^2 points and is smooth away from the origin. By Hensel's lemma each of the (q^2-1) smooth points lifts to $q^{2(m+1)}$ solutions of $x^2+yz = 0$ over $\mathfrak{o}/\pi^{m+2}\mathfrak{o}$. All the other solutions over $\mathfrak{o}/\pi^{m+2}\mathfrak{o}$ are of the form $(\pi x, \pi y, \pi z)$, thus x, y, z are perturbations in $\mathfrak{o}/\pi^{m+1}\mathfrak{o}$ of solutions modulo π^m and the claim follows.

The argument for part (2) is similar, but as the $\mathbb{F}_q$-points of the variety defined by $x^2 + \pi yz \equiv x^2$ are all non-smooth, we consider higher levels. The set of solutions of $x^2 + \pi yz = 0$ in $(\mathfrak{o}/\pi^{m+2}\mathfrak{o})^3$ is a subset of the set

$$\{(\pi x, y, z) \in (\mathfrak{o}/\pi^{m+2}\mathfrak{o})^3 \mid \text{exactly one of } y \text{ or } z \text{ is a unit}\}$$
$$\cup \{(\pi x, \pi y, \pi z) \in (\mathfrak{o}/\pi^{m+2}\mathfrak{o})^3 \mid x^2 + \pi yz \equiv 0 \mod \pi^m\}.$$

The number of solutions of the second type is $q^3f(0,1,m)$. For the first type, assuming that z is a unit and $\pi \mid y$, we are left to solve $\pi\tilde{x}^2 + \tilde{y}z \equiv 0 \mod \pi^m$, where $x = \pi\tilde{x}$ and $y = \pi\tilde{y}$. Note that $\tilde{y}$ is completely determined by $\tilde{x}, z$. Counting in redundancy from the reduction, we find $(q-1)q^{2m+3}$ solutions. By symmetry, the total number of solutions for this type is $2(q-1)q^{2m+3}$.

To prove part (3) consider the equation $x^2+\pi^{\beta+2}yz=0$ over $\mathfrak{o}/\pi^{m+2}\mathfrak{o}$. Note that a triple (x,y,z) is a solution if and only if $x=\pi\tilde{x}$, and the triple $(\tilde{x},y,z)$ is a solution of the equation $\pi^2(\tilde{x}^2+\pi^\beta yz)=0$ over $\mathfrak{o}/\pi^{m+2}\mathfrak{o}$. Thus

$$f(0,\beta+2,m+2)=q^5 f(0,\beta,m),$$

where the factor q^5 comes from the redundancy arising from the reduction to $\tilde{x}^2+\pi^\beta yz\equiv 0 \mod \pi^m$.

For part (4), to solve the equation $\pi^{\alpha+1}x^2+yz=0$ over $\mathfrak{o}/p^{m+1}\mathfrak{o}$ we consider two cases: that y is divisible by π or that y is a unit. Using arguments similar to those above, we find in the first case $q^2 f(\alpha,0,m)$ solutions and in the second $q(q-1)q^{2m}$ solutions. ∎

Proof of Proposition 5.2. We first compute $F_{0,0}(t)$. We multiply both sides of equation (1) in Lemma 5.3 by t^{m+2} and sum over the non-negative integers to obtain

$$\sum_{m=0}^{\infty} f(0,0,m+2)t^{m+2}=q^2(q^2-1)\sum_{m=0}^{\infty} q^{2m}t^{m+2}+q^3\sum_{m=0}^{\infty} f(0,0,m)t^{m+2}.$$

Using the fact that $f(0,0,0)=1$ and $f(0,0,1)=q^2$ we get

$$F_{0,0}(t)-1-q^2t=\frac{q^2(q^2-1)t^2}{1-q^2t}+q^3t^2F_{0,0}(t),$$

which implies the formula for $F_{0,0}(t)$. The derivation of $F_{0,1}(t)$ is similar.

To prove part (i) we multiply both sides of equation (4) in Lemma 5.3 by t^{m+1} and sum over the non-negative integers. This gives

$$\underbrace{\sum_{m=0}^{\infty} f(\alpha+1,0,m+1)t^{m+1}}_{F_{\alpha+1,0}(t)-1}=q(q-1)\sum_{m=0}^{\infty} q^{2m}t^{m+1}+\underbrace{q^2\sum_{m=0}^{\infty} f(\alpha,0,m)t^{m+1}}_{q^2tF_{\alpha,0}(t)},$$

and thus yields the recurrence

$$F_{\alpha+1,0}(t)=\frac{1-qt}{1-q^2t}+q^2tF_{\alpha,0}(t).$$

A recurrence of this form, namely, $A_{\alpha+1}=d+cA_\alpha$ ($\alpha\in\mathbb{N}_0$), has the following solution

$$A_\alpha=d\,\frac{1-c^\alpha}{1-c}+c^\alpha A_0,\quad \alpha\in\mathbb{N}_0, \tag{5.2}$$

which implies part (i) of the proposition.

Similarly, to prove part (ii) we multiply both sides of equation (3) in Lemma 5.3 by t^{m+2} and sum over the non-negative integers:

$$\begin{aligned} F_{0,\alpha+2}(t) - 1 - q^2t &= \sum_{m=0}^{\infty} f(0,\alpha+2,m+2)t^{m+2} \\ &= q^5 \sum_{m=0}^{\infty} f(0,\alpha,m)t^{m+2} = q^5t^2F_{0,\alpha}(t). \end{aligned}$$

We get the recurrence relation

$$F_{0,\alpha+2}(t) = 1 + q^2t + q^5t^2F_{0,\alpha}(t).$$

This is solved separately for even and odd α, via (5.2), giving

$$F_{0,\alpha}(t) = (1+q^2t)\frac{1-q^{5\lfloor \frac{\alpha}{2} \rfloor}t^{2\lfloor \frac{\alpha}{2} \rfloor}}{1-q^5t^2} + (q^5t^2)^{\lfloor \alpha/2 \rfloor}F_{0,\overline{\alpha}}(t). \qquad \blacksquare$$

We need to pin down the variant $F^\star_{0,\alpha}(t)$ of $F_{0,\alpha}(t)$, which was introduced in Definition 5.1.

LEMMA 5.4: *For $\alpha \in \mathbb{N}_0$, set $\overline{\alpha} = 0$ for α even and $\overline{\alpha} = 1$ for α odd. Then*

$$\begin{aligned} F^\star_{0,\alpha}(t) &= (q^5t^2)^{\lfloor \alpha/2 \rfloor}F^\star_{0,\overline{\alpha}}(t) \\ &= \begin{cases} q^{\frac{5\alpha}{2}}t^\alpha F_{0,0}(t) = q^{\frac{5\alpha}{2}}t^\alpha \frac{1-q^2t^2}{(1-q^2t)(1-q^3t^2)} & \text{for } \alpha \text{ even,} \\ q^{\frac{5(\alpha-1)}{2}}t^{\alpha-1}(F_{0,1}(t)-1) = q^{\frac{5\alpha-1}{2}}t^\alpha \frac{(1-qt)(1+q^2t)}{(1-q^2t)(1-q^3t^2)} & \text{for } \alpha \text{ odd.} \end{cases} \end{aligned}$$

Furthermore, employing another indeterminate Y, we have

$$\sum_{\alpha=0}^{\infty} Y^\alpha F^\star_{0,\alpha}(t) = \frac{(1-qt)(1+qt+Yq^2t(1+q^2t))}{(1-q^5t^2Y^2)(1-p^2t)(1-q^3t^2)}.$$

Proof. Multiplying both sides of equation (4) in Lemma 5.3 by t^{m+2} and summing over $m \geq \alpha$, we obtain

$$F^\star_{0,\alpha+2}(t) = \sum_{m=\alpha}^{\infty} f(0,\alpha+2,m+2)t^{m+2} = q^5t^2 \sum_{m=\alpha}^{\infty} f(0,\alpha,m)t^m = q^5t^2F^\star_{0,\alpha}(t).$$

Writing $\alpha = 2j + \varepsilon$ with $\varepsilon \in \{0,1\}$, we deduce that

$$F^\star_{0,2j+\varepsilon}(t) = q^{5j}t^{2j}F^\star_{0,\varepsilon}(t).$$

By substituting $F^\star_{0,0}(t) = F_{0,0}(t)$ and $F^\star_{0,1}(t) = F_{0,1}(t) - 1$ we arrive at the desired formula.

The last part follows by substituting the formulae obtained into

$$\sum_{\alpha=0}^{\infty} Y^{\alpha} F^{\star}_{0,\alpha}(t) = \sum_{j=0}^{\infty} Y^{2j} F^{\star}_{0,2j}(t) + \sum_{j=0}^{\infty} Y^{2j+1} F^{\star}_{0,2j+1}(t). \qquad \blacksquare$$

5.2. Applying a p-adic Bruhat decomposition. We now turn our attention to the pro-isomorphic zeta function of the D^*-group $\Gamma = \Gamma_{t^3}$ of Hirsch length 8, defined in (1.3), and we prove Theorem 1.3.

Proposition 3.1 shows that $\zeta^{\wedge}_{\Gamma,p}(s) = \zeta^{\wedge}_{L,p}(s)$ for all primes p, where L is the $\mathbb{Z}$-Lie lattice associated to Γ. In our setting, L is the $\mathbb{Q}$-indecomposable D^*-Lie lattice L of $\mathbb{Z}$-rank 8, defined by (2.2) with respect to the $\mathbb{Z}$-basis $\mathcal{S}$, where

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is the companion matrix of the primary polynomial $\Delta_K = t^2$. We consider the algebraic automorphism group $\mathbf{G} = \mathbf{Aut}(L)$, with respect to the $\mathbb{Z}$-basis $\mathcal{S}^* = (x_1, y_3,\, x_2, y_2,\, x_3, y_1,\, z_2, z_1)$ as in Corollary 2.7 and Example 2.9.

Let p be a prime; we will set about calculating the local pro-isomorphic zeta function $\zeta^{\wedge}_{L,p}(s)$. In the notation of Section 3, we set

$$U_1 = \mathrm{span}_{\mathbb{Q}_p}\{x_1, y_3,\, x_2, y_2,\, x_3, y_1\} \quad \text{and} \quad U_2 = \mathrm{span}_{\mathbb{Q}_p}\{z_2, z_1\}.$$

We write

$$G = \mathbf{G}(\mathbb{Q}_p), \quad H = \mathbf{H}(\mathbb{Q}_p), \quad N = \mathbf{N}(\mathbb{Q}_p);$$

these groups act on $V = {}_{\mathbb{Q}_p}L = U_1 \oplus U_2$. In accordance with Corollary 2.7, the elements of the reductive subgroup H can be written in the form

$$\left(\begin{array}{ccc|cc} \nu^{-1}A & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & \nu A & 0 & 0 \\ \hline 0 & 0 & 0 & \nu^{-1}\det A & 0 \\ 0 & 0 & 0 & 0 & \det A \end{array}\right), \tag{5.3}$$

where $(A, \nu) \in \mathsf{GL}_2(\mathbb{Q}_p) \times \mathsf{GL}_1(\mathbb{Q}_p)$; observe that we have performed a routine reparametrisation $\nu \mapsto \nu^{-1}$ and $A \mapsto \nu A$: for our computation of ϑ we prefer to have the powers of ν appearing along the diagonal to be 'small'.

The description of the unipotent radical given in Example 2.9 shows that elements of N are of the form

$$u(B,C) = \left(\begin{array}{ccc|cc} \mathrm{I}_2 & B & C & * & * \\ 0 & \mathrm{I}_2 & B+\lambda \mathrm{I}_2 & * & * \\ 0 & 0 & \mathrm{I}_2 & * & * \\ \hline 0 & 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \tag{5.4}$$

where $B, C \in \mathsf{M}_2(\mathbb{Q}_p)$ and $\lambda \in \mathbb{Q}_p$ with $\mathrm{tr}(B) = 0$, $\mathrm{tr}(C)+\det(B) = 0$, and there are arbitrary entries in the positions marked $*$. As explained in Section 3, we can utilise Proposition 3.3 and Theorem 3.6 to compute $\zeta^{\wedge}_{L,p}(s)$ via an integral over H^+.

We now return to our coarse decomposition and set about calculating the functions ϑ_0, ϑ_1 defined in Section 3; we refer to Section 3.2 for definitions of N_1, μ_{N/N_1} and μ_{N_1}. Noting that $N_1 \cong \mathbb{Q}_p^8$, we obtain for $h \in H^+$ of the form (5.3)

$$\vartheta_1(h) = |\nu^{-1} \det A^2|_p^{-6} = |\det A|_p^{-12} |\nu|_p^6,$$

hence $\vartheta(h) = \vartheta_0(h)\vartheta_1(h) = \vartheta_0(h)|\det A|_p^{-12}|\nu|_p^6$. We defer until the next section a calculation of ϑ_0, since this is the most involved and lengthy aspect of the analysis.

We observe that the morphism

$$\varrho\colon \dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1 \to \mathbf{H}, \quad (A,\nu) \mapsto \mathrm{diag}(\nu^{-1}A, A, \nu A,\ \nu^{-1}\det A, \det A)$$

induces a measure-preserving isomorphism $\dot{H} = \dot{\mathbf{H}}(\mathbb{Q}_p) \to H$ such that

$$H^+\varrho^{-1} = \{(A,\nu) \mid v_p(A) \geq 0 \text{ and } v_p(A) - |v_p(\nu)| \geq 0\},$$

where v_p is defined as in Example 3.8 and in Section 4. Thus we obtain

$$\zeta^{\wedge}_{L,p}(s) = \int_{\substack{(A,\nu)\in\dot{H} \\ \text{with } v_p(A)\geq 0, \\ v_p(A)+|v_p(\nu)|\geq 0}} |\det A|_p^{5s-12}\, |\nu|_p^{-s+6} \vartheta_0((A,\nu)^{\varrho})\, \mathrm{d}\mu_p(A,\nu). \tag{5.5}$$

For convenience, we consider $\dot{\mathbf{H}} = \mathsf{GL}_2 \times \mathsf{GL}_1$ as a subgroup of GL_3, embedded as block matrices via $(A,\nu) \mapsto \mathrm{diag}(A,\nu)$. In particular,

$$T = \mathbf{T}(\mathbb{Q}_p) = \{\mathrm{diag}(\lambda_1, \lambda_2, \nu) \mid \lambda_1, \lambda_2, \nu \in \mathbb{Q}_p^{\times}\}$$

is a maximal torus in $\dot{H}$.

By Proposition 3.7 we obtain

$$\zeta^{\wedge}_{L,p}(s) = \sum_{w\in W} p^{-\operatorname{len}(w)} \sum_{\xi\in w\Xi^+_w} |\alpha(\xi(\pi))|_p^{-1}\ |\det(\xi(\pi)^{\varrho})|_p^{s}\ \vartheta(\xi(\pi)^{\varrho}),$$

where we choose $\alpha \in \operatorname{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}}), \alpha(\operatorname{diag}(\lambda_1, \lambda_2, \nu)) = \lambda_1\lambda_2^{-1}$ as the single positive root, and we have

$$w\Xi^+_w = \{\xi \in \Xi^+ \mid \alpha(\xi(\pi)) \in \mathbb{Z}_p, \text{ and } \alpha(\xi(\pi)) \in p\mathbb{Z}_p \text{ if } w = w_0\},$$

where the Weyl group is $W = \{1, w_0\}$. In order to describe the set $w\Xi^+_w$ we will need to consider dominant weights for the contragredient representation, following [14]. These are given by

$$\omega_1^{-1}(h) = \lambda_2\nu^{-1}, \quad \omega_2^{-1}(h) = \lambda_2, \quad \omega_3^{-1}(h) = \lambda_2\nu,$$
$$\omega_4^{-1}(h) = \lambda_1\lambda_2\nu^{-1}, \quad \omega_5^{-1}(h) = \lambda_1\lambda_2$$

for $h = \operatorname{diag}(\lambda_1, \lambda_2, \nu) \in T$. It follows that $\alpha, \omega_1^{-1}, \omega_2^{-1}$ form a $\mathbb{Z}$-basis for $\operatorname{Hom}(\mathbf{T}, \mathsf{G}_{\mathrm{m}})$. Unlike the situation in Section 4, the $\mathbb{N}_0$-span of these three dominant weights does not contain all the weights of ϱ. In the current situation an element $h \in T$ is integral if and only if $\alpha(h), \omega_1^{-1}(h), \omega_2^{-1}(h), \omega_3^{-1}(h)$ all lie in $\mathbb{Z}_p$. Note that $\omega_3^{-1} = \omega_1\omega_2^{-2}$. We rewrite $\alpha_1 = \alpha$, $\alpha_2 = \omega_1^{-1}$, $\alpha_3 = \omega_2^{-1}$ and seek a dual basis, namely elements $\xi_1, \xi_2, \xi_3 \in \Xi$ such that

$$\langle \alpha_i, \xi_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

A routine calculation shows that the following elements suffice:

$$\xi_1(\tau) = (\tau, 1, 1), \quad \xi_2(\tau) = (1, 1, \tau^{-1}), \quad \xi_3(\tau) = (\tau, \tau, \tau) \quad \text{for } \tau \in \mathbb{Q}_p^{\times}.$$

A general element of Ξ has the form $\xi_{\mathbf{e}} = \xi_1^{e_1}\xi_2^{e_2}\xi_3^{e_3}$ with $\mathbf{e} = (e_1, e_2, e_3) \in \mathbb{Z}^3$ and then

$$\xi_{\mathbf{e}}(\pi) = \operatorname{diag}(\pi^{e_1+e_3}, \pi^{e_3}, \pi^{e_3-e_2}). \tag{5.6}$$

Hence

$$\xi_{\mathbf{e}}(\pi)^{\varrho} = \operatorname{diag}(\pi^{e_1+e_2}, \pi^{e_2}, \pi^{e_1+e_3}, \pi^{e_3}, \pi^{e_1-e_2+2e_3}, \pi^{-e_2+2e_3}, \pi^{e_1+e_2+e_3}, \pi^{e_1+2e_3})$$

and we read off

$$|\det \xi_{\mathbf{e}}(\pi)^{\varrho}|_p^{s} = p^{-(5e_1+e_2+9e_3)s}, \quad \vartheta_1(\xi_{\mathbf{e}}(\pi)^{\varrho}) = p^{12e_1+6e_2+18e_3}.$$

Note that $|\alpha(\xi_{\mathbf{e}}(\pi))|^{-1} = p^{\langle\alpha,\xi_{\mathbf{e}}\rangle} = p^{\langle\alpha_1,\xi_{\mathbf{e}}\rangle} = p^{e_1}$ and we can rewrite

$$\begin{aligned} w\Xi_w^+ &= \{\xi\in\Xi \mid \langle\alpha_i,\xi\rangle\geq 0 \text{ for } i\in\{1,2,3\};\ \langle\omega_3^{-1},\xi\rangle\geq 0, \text{ and } \langle\alpha_1,\xi\rangle>0 \text{ if } w=w_0\} \\ &= \{\xi_{\mathbf{e}} \mid e_i\geq 0 \text{ for } i\in\{1,2,3\};\ 2e_3\geq e_2, \text{ and } e_1>0 \text{ if } w=w_0\}, \end{aligned}$$

since $\alpha_1 \in w(\Phi^-)$ if and only if $w \neq 1$ and $\omega_3^{-1} = \omega_1\omega_2^{-2} = \alpha_2^{-1}\alpha_3^2$. Writing

$$\mathcal{C} = \{\mathbf{e}\in\mathbb{N}_0^3 \mid 2e_3\geq e_2\} \tag{5.7}$$

we obtain

$$\begin{aligned} &\mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,p}(s) \\ &\quad= \sum_{w\in W} p^{-\operatorname{len}(w)} \sum_{\xi\in w\Xi_w^+} p^{\langle\alpha,\xi\rangle}\,|\det\xi(\pi)^{\varrho}|_p^s\,\vartheta(\xi(\pi)^{\varrho}) \\ &\quad= \sum_{w\in W} p^{-\operatorname{len}(w)} \sum_{\substack{\mathbf{e}\in\mathcal{C} \text{ with} \\ e_1>0 \text{ if } w\neq 1}} p^{(13-5s)e_1+(6-s)e_2+(18-9s)e_3}\vartheta_0(\xi_{\mathbf{e}}(\pi)^{\varrho}). \end{aligned} \tag{5.8}$$

5.3. Determining the function ϑ_0. In view of (5.3) and (5.8), we need only compute ϑ_0 for elements of H of a rather special form; for $n,m,k\in\mathbb{Z}$ we set

$$\begin{aligned} \vartheta_0(\pi^n,\pi^m,\pi^k) &= \vartheta_0(\operatorname{diag}(\pi^n,\pi^m,\pi^k)^{\varrho}) \\ &= \vartheta_0(\operatorname{diag}(\pi^{n-k},\pi^{m-k},\,\pi^n,\pi^m,\,\pi^{n+k},\pi^{m+k},\,\pi^{m+n-k},\pi^{m+n})), \end{aligned}$$

where the first expression is a mild, but convenient abuse of notation. Recall that we could choose $\pi = p$, but prefer to make clear the different roles played by π and p. This is beneficial also with a view toward the more general situation considered in Section 7; we refrain from generalising all the notation in the current section as we did in Section 5.1, but explain in Remark 5.7 how one particular step carries over. We assume throughout that $n\geq m$ since this is the only case of interest to us; see (5.6). Write $l = n-m\in\mathbb{N}_0$, and recall from Definition 5.1 with $\mathfrak{o} = \mathbb{Z}_p$ that

$$f(\alpha,\beta,m) = \#\{(x,y,z)\in(\mathbb{Z}_p/\pi^m\mathbb{Z}_p)^3 \mid \pi^\alpha x^2+\pi^\beta yz = 0\}$$

for $\alpha,\beta,m\in\mathbb{N}_0$.

LEMMA 5.5: *For $n, m, k \in \mathbb{Z}$ with $l = n - m \geq 0$, we have*

$$\vartheta_0(\pi^n, \pi^m, \pi^k) = p^{4k+3m+n}\, \tilde{\vartheta}(\pi^n, \pi^m, \pi^k),$$

where $\tilde{\vartheta}(\pi^n, \pi^m, \pi^k)$ equals

$$\begin{cases} p^{3k+l} f(l, 0, m-k) & \text{if } k \geq 0 \qquad \text{(Case 1)}, \\ p^{-k+l} f(2k+l, 0, m+k) & \text{if } \max\{-m,-l\} \leq k < 0 \text{ and } 2k+l \geq 0 \qquad \text{(Case 2a)}, \\ p^{5k+4l} f(0, -2k-l, m-k-l) & \text{if } \max\{-m,-l\} \leq k < 0 \text{ and } 2k+l < 0 \qquad \text{(Case 2b)}, \\ p^{-l} f(0, l, n+k) & \text{if } -m \leq k < -l \qquad \text{(Case 3)}. \end{cases}$$

Remark 5.6: It follows from Lemma 5.5 that $\vartheta\colon H \to \mathbb{R}_{>0}$ is not a character. For instance,

$$\tilde{\vartheta}(\pi^2, \pi^2, 1) = f(0,0,2) = p^4 + p^3 - p^2 \neq p^4 = f(0,0,1)^2 = \tilde{\vartheta}(\pi, \pi, 1)^2.$$

In fact, this calculation shows that the lifting condition [14, Assumption 2.3] fails for all primes p. Suppose that the lifting condition were to hold. By [4, Lem. 3.12], it would follow that ϑ is a character on subsets of a maximal torus of H with a designated ordering of valuations along the diagonal. It is readily seen that the elements $\mathrm{diag}(\pi^2, \pi^2, 1)^\varrho$, $\mathrm{diag}(\pi, \pi, 1)^\varrho$ belong to such a subset.

Proof of Lemma 5.5. We consider the action of a diagonal element

$$h = \mathrm{diag}(\pi^{n-k}, \pi^{m-k}, \pi^n, \pi^m, \pi^{n+k}, \pi^{m+k}, \pi^{m+n-k}, \pi^{m+n})$$

on an element

$$u = \left(\begin{array}{ccc|cc} \mathrm{I}_2 & \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) & \left(\begin{smallmatrix} d & e \\ f & a^2+bc-d \end{smallmatrix}\right) & * & * \\ 0 & \mathrm{I}_2 & \left(\begin{smallmatrix} \lambda+a & b \\ c & \lambda-a \end{smallmatrix}\right) & * & * \\ 0 & 0 & \mathrm{I}_2 & * & * \\ \hline 0 & 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \tag{5.9}$$

the latter being an explicit parametrisation of (5.4). The situation of interest to us, i.e., when h is integral, is equivalent to the conditions $n \geq m \geq |k|$. We

obtain the following necessary and sufficient conditions for uh to be integral:

$$v_p(a) \geq -m, \tag{5.10}$$

$$v_p(b) \geq -m + \max\{0, -k\}, \tag{5.11}$$

$$v_p(c) \geq -n + \max\{0, -k\}, \tag{5.12}$$

$$v_p(d) \geq -n - k, \tag{5.13}$$

$$v_p(e) \geq -m - k, \tag{5.14}$$

$$v_p(f) \geq -n - k, \tag{5.15}$$

$$v_p(a^2 + bc - d) \geq -m - k, \tag{5.16}$$

$$v_p(\lambda + a) \geq -n - k, \tag{5.17}$$

$$v_p(\lambda - a) \geq -m - k, \tag{5.18}$$

$$v_p(\lambda) \geq -m - n. \tag{5.19}$$

Condition (5.19) is implied by conditions (5.10) and (5.18); it is therefore redundant. One readily sees the following equivalences:

$$(5.13):\ v_p(d) \geq -n-k \iff v_p(a^2+bc) \geq -n-k, \quad \text{if (5.16) holds;}$$

$$(5.17):\ v_p(\lambda+a) \geq -n-k \iff v_p(2a) \geq -n-k, \quad \text{if (5.18) holds;}$$

so we may replace (5.13) and (5.17) respectively by

$$v_p(a^2 + bc) \geq -n - k, \tag{5.13}'$$

$$v_p(a) \geq -n - k - \delta, \tag{5.17}'$$

where $\delta = v_p(2) \in \{0, 1\}$ takes the value 1 for $p = 2$ and the value 0 otherwise.

In our calculation we use the fact that the measure μ_{N/N_1} may be treated as an additive measure on the parameter space $\mathbb{Q}_p^7$ with $(N/N_1)(\mathbb{Z}_p)$ corresponding to $\mathbb{Z}_p^7$. Indeed, using the notation introduced in (5.4), we see that the map

$$\mathsf{M}_2(\mathbb{Q}_p) \times \mathsf{sl}_2(\mathbb{Q}_p) \to N/N_1(\mathbb{Q}_p), \quad (X, Y) \mapsto u(X, Y)$$

is a homeomorphism. The claim thus follows from [26, Thm. 8.32] and the fact that the groups involved are unimodular.

For fixed parameters $(a, b, c) \in \mathbb{Q}_p^3$, we obtain

$$\mu_{\mathbb{Q}_p^4}\{(d, e, f, \lambda) \in \mathbb{Q}_p^4 \mid (5.14), (5.15), (5.16), (5.18) \text{ hold}\} = p^{3m+n+4k}.$$

It follows that $\vartheta_0(\pi^n, \pi^m, \pi^k) = p^{3m+n+4k}\, \tilde{\vartheta}(\pi^n, \pi^m, \pi^k)$, where

$$\tilde{\vartheta}(\pi^n, \pi^m, \pi^k) = \mu_{\mathbb{Q}_p^3}\{(a, b, c) \in \mathbb{Q}_p^3 \mid (5.10), (5.11), (5.12), (5.13)', (5.17)' \text{ hold}\}.$$

For convenience, we summarise the conditions (5.10), (5.11), (5.12), $(5.13)'$ and $(5.17)'$:

$$(\dagger)\quad \begin{aligned} &v_p(a) \geq \max\{-m, -n-k-\delta\}, && v_p(b) \geq -m + \max\{0, -k\},\\ &v_p(c) \geq -n + \max\{0, -k\}, && v_p(a^2+bc) \geq -n-k. \end{aligned}$$

The next step is to show that we can drop δ, even for $p = 2$. Suppose for a contradiction, that there are $a, b, c \in \mathbb{Q}_p$ satisfying (†) and such that $v_p(a) = -n-k-1 \geq -m$; in particular, $k < 0$. Then

$$v_p(a^2) = -2n-2k-2 < -n-k$$

and we conclude from $(5.13)'$ that $v_p(bc) = v_p(a^2) = -2n-2k-2$. On the other hand (5.11) and (5.12) yield $v_p(bc) \geq -n-m-2k$. This gives $-2n-2k-2 \geq -n-m-2k$, hence $m-2 \geq n$, a contradiction.

Remark 5.7: The last consideration carries through also in a more general setting, considered in Section 7. If we work over a compact discrete valuation ring $\mathfrak{o}$ with valuation $v_\wp$, replacing $\mathbb{Z}_p$ with valuation v_p, then $\delta = v_\wp(2)$. If $\mathfrak{o}$ has residue characteristic 2 this is the absolute ramification index of $\mathfrak{o}$, and the assumption $v_\wp(a) = -n-k-\bar{\delta} \geq -m$ with $\bar{\delta} \in \{1, \ldots, \delta\}$ leads again to a contradiction.

Thus we can work with the simpler set of conditions

$$(\ddagger)\quad \begin{aligned} &v_p(a) \geq \max\{-m, -n-k\}, && v_p(b) \geq -m + \max\{0, -k\},\\ &v_p(c) \geq -n + \max\{0, -k\}, && v_p(a^2+bc) \geq -n-k. \end{aligned}$$

We perform a change of variables $\mathbb{Q}_p{}^3 \to \mathbb{Q}_p^3$ by

$$(a, b, c) \mapsto (x, y, z) = (ap^{\min\{m, n+k\}}, bp^{m+\min\{0,k\}}, cp^{n+\min\{0,k\}}).$$

The new variables are all unconstrained elements of $\mathbb{Z}_p$, and the change of variables introduces a Jacobian equal to

$$p^{\min\{m,n+k\}+m+n+\min\{0,2k\}}.$$

It follows that

$$\begin{aligned} &\tilde{\vartheta}(\pi^n, \pi^m, \pi^k)\\ &= \mu_{\mathbb{Q}_p^3}\{(a,b,c) \in \mathbb{Q}_p^3 \mid (\ddagger) \text{ holds}\}\\ &= p^{\min\{m,n+k\}+m+n+\min\{0,2k\}}\\ &\quad \cdot \mu_{\mathbb{Z}_p^3}\{(x,y,z) \in \mathbb{Z}_p^3 \mid p^{-2\min\{m,n+k\}}x^2 + p^{-m-n-\min\{0,2k\}}yz \equiv 0 \bmod p^{-n-k}\}. \end{aligned}$$

Lemma 5.5 now follows immediately by specialising to the four cases. ∎

In order to continue the calculation paused at (5.8), we recall that

$$\xi_{\mathbf{e}} = \xi_1^{e_1}\xi_2^{e_2}\xi_3^{e_3}$$

and, setting

$$n_{\mathbf{e}} = e_1 + e_3, \quad m_{\mathbf{e}} = e_3, \quad k_{\mathbf{e}} = e_3 - e_2, \quad \text{thus} \quad l_{\mathbf{e}} = n_{\mathbf{e}} - m_{\mathbf{e}} = e_1, \tag{5.20}$$

we see from (5.6) that $\xi_{\mathbf{e}}(\pi) = \operatorname{diag}(\pi^{e_1+e_3}, \pi^{e_3}, \pi^{e_3-e_2}) = \operatorname{diag}(\pi^{n_{\mathbf{e}}}, \pi^{m_{\mathbf{e}}}, \pi^{k_{\mathbf{e}}})$. Applying Lemma 5.5 and using (5.20) to resubstitute, we obtain

$$\vartheta_0(\xi_{\mathbf{e}}(\pi)^{\varrho}) = p^{e_1-4e_2+8e_3}\,\tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)), \tag{5.21}$$

where

$$\tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)) = \begin{cases} p^{e_1-3e_2+3e_3} f(e_1, 0, e_2) & \\ \qquad\qquad \text{if } e_2 \le e_3 \text{ (Case 1)}, & \\ p^{e_1+e_2-e_3} f(e_1 - 2e_2 + 2e_3, 0, -e_2 + 2e_3) & \\ \qquad \text{if } e_3 < e_2 \le e_3 + \min\{e_1, e_3\} \text{ and } 2e_2 \le e_1 + 2e_3 \text{ (Case 2a)}, & \\ p^{4e_1-5e_2+5e_3} f(0, -e_1 + 2e_2 - 2e_3, -e_1 + e_2) & \\ \qquad \text{if } e_3 < e_2 \le e_3 + \min\{e_1, e_3\} \text{ and } e_1 + 2e_3 < 2e_2 \text{ (Case 2b)}, & \\ p^{-e_1} f(0, e_1, e_1 - e_2 + 2e_3) & \\ \qquad\qquad \text{if } e_1 + e_3 < e_2 \le 2e_3 \text{ (Case 3)}. & \end{cases}$$

Referring to (5.8), we obtain

$$\mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,p}(s) = \sum_{w \in W} p^{-\operatorname{len}(w)} \sum_{\substack{\mathbf{e} \in \mathcal{C} \text{ with} \\ e_1 > 0 \text{ if } w \neq 1}} X_1^{e_1} X_2^{e_2} X_3^{e_3}\, \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)), \tag{5.22}$$

where

$$X_1 = p^{14-5s}, \quad X_2 = p^{2-s}, \quad X_3 = p^{26-9s}.$$

5.4. Decomposing the polyhedral cone. In preparation of the final stage of the calculation, we consider the following subsets of the 'integral' cone $\mathcal{C}$ introduced in (5.7); each subset is, in fact, a submonoid of $\mathbb{N}_0^3$. Refer to Figure 1 for a pictorial illustration.

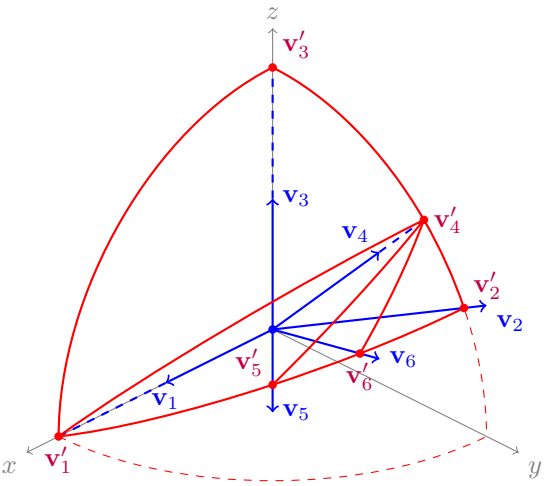

Figure 1. Decomposition of the cone $\mathcal{C}$.

Definition 5.8: Write

$$\mathbf{v}_1 = (1,0,0), \quad \mathbf{v}_2 = (0,2,1), \quad \mathbf{v}_3 = (0,0,1),$$
$$\mathbf{v}_4 = (0,1,1), \quad \mathbf{v}_5 = (2,2,1), \quad \mathbf{v}_6 = (1,2,1),$$

and set

$$\begin{aligned}
\mathcal{C}_{ijk} &= \operatorname{span}_{\mathbb{N}_0}\{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k\} && \text{for } 1 \le i,j,k \le 6,\\
\mathcal{C}_{ijk^+} &= \operatorname{span}_{\mathbb{N}_0}\{\mathbf{v}_i, \mathbf{v}_j\} + \mathbb{N}\mathbf{v}_k && \text{for } 1 \le i,j,k \le 6,\\
\mathcal{C}_{ij} &= \operatorname{span}_{\mathbb{N}_0}\{\mathbf{v}_i, \mathbf{v}_j\} && \text{for } 1 \le i,j \le 6,\\
\mathcal{C}_{ij^+} &= \mathbb{N}_0\mathbf{v}_i + \mathbb{N}\mathbf{v}_j && \text{for } 1 \le i,j \le 6,\\
\mathcal{C}_*^0 &= \{(e_1,e_2,e_3) \in \mathcal{C}_* \mid e_1 > 0\} && \text{for any (possibly empty) index } *.
\end{aligned}$$

Observation 5.9: The elements $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the completely fundamental elements of $\mathcal{C}$, while $\mathbf{v}_4 = \frac{1}{2}(\mathbf{v}_2 + \mathbf{v}_3)$ is merely fundamental; compare with [31, Chap. I]. A routine verification shows that

$$\begin{aligned}
\mathcal{C}_{134} &= \{(e_1,e_2,e_3) \in \mathcal{C} \mid e_2 \le e_3\},\\
\mathcal{C}_{145^+} &= \{(e_1,e_2,e_3) \in \mathcal{C} \mid e_3 < e_2 \le e_3 + \min\{e_1,e_3\},\ 2e_2 \le e_1 + 2e_3\},\\
\mathcal{C}_{456^+} &= \{(e_1,e_2,e_3) \in \mathcal{C} \mid e_3 < e_2 \le e_3 + \min\{e_1,e_3\},\ e_1 + 2e_3 < 2e_2\},\\
\mathcal{C}_{462^+} &= \{(e_1,e_2,e_3) \in \mathcal{C} \mid e_1 + e_3 < e_2 \le 2e_3\};
\end{aligned}$$

hence the sets $\mathcal{C}_{134}$, $\mathcal{C}_{145^+}$, $\mathcal{C}_{456^+}$, $\mathcal{C}_{462^+}$ correspond precisely to Cases 1, 2a, 2b and 3 in Lemma 5.5; compare with (5.21).

The following decompositions are easily verified:

$$\begin{aligned} &\mathcal{C} = \mathcal{C}_{134} \cup \mathcal{C}_{145^+} \cup \mathcal{C}_{456^+} \cup \mathcal{C}_{462^+}, \quad && \mathcal{C}^0 = \mathcal{C}^0_{134} \cup \mathcal{C}^0_{145^+} \cup \mathcal{C}^0_{456^+} \cup \mathcal{C}^0_{462^+}, \\ &\mathcal{C}_{234} = \mathcal{C}_{34} \cup \mathcal{C}_{42^+}, && \mathcal{C} = \mathcal{C}^0 \cup \mathcal{C}_{234}. \end{aligned} \tag{5.23}$$

For convenience, for a subset $\mathcal{C}_{ijk} \subseteq \mathcal{C}$ write

$$Z_{ijk}(s) = \sum_{\mathbf{e} \in \mathcal{C}_{ijk}} X_1^{e_1} X_2^{e_2} X_3^{e_3} \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi))$$

and adopt a similar shorthand notation for subsets of the form $\mathcal{C}_{ijk^+}, \mathcal{C}_{ij}, \mathcal{C}_{ij^+}$. From (5.22) and (5.23) we deduce that

$$\begin{aligned} \mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,p}(s) &= \sum_{\mathbf{e} \in \mathcal{C}} X_1^{e_1} X_2^{e_2} X_3^{e_3} \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)) + p^{-1} \sum_{\mathbf{e} \in \mathcal{C}^0} X_1^{e_1} X_2^{e_2} X_3^{e_3} \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)) \\ &= (1+p^{-1}) \sum_{\mathbf{e} \in \mathcal{C}} X_1^{e_1} X_2^{e_2} X_3^{e_3} \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)) - p^{-1} Z_{234}(s) \\ &= (1+p^{-1})(Z_{134}(s) + Z_{145^+}(s) + Z_{456^+}(s) + Z_{462^+}(s)) \\ &\quad - p^{-1}(Z_{34}(s) + Z_{42^+}(s)). \end{aligned} \tag{5.24}$$

LEMMA 5.10: *Referring to Definition 5.1, we have*

$$\begin{aligned} Z_{134}(s) &= \frac{1}{1-p^3X_3} \sum_{i=0}^{\infty} (pX_1)^i F_{i,0}(X_2X_3), \\ Z_{145^+}(s) &= \frac{p^3X_1^2X_2^2X_3}{1-p^3X_1^2X_2^2X_3} \sum_{i=0}^{\infty} (pX_1)^i F_{i,0}(X_2X_3), \\ Z_{456^+}(s) &= \frac{1}{1-p^3X_1^2X_2^2X_3} \sum_{i=1}^{\infty} (p^{-1}X_1X_2)^i F^*_{0,i}(X_2X_3), \\ Z_{462^+}(s) &= \frac{X_2^2X_3}{1-X_2^2X_3} \sum_{i=0}^{\infty} (p^{-1}X_1X_2)^i F^*_{0,i}(X_2X_3), \\ Z_{34}(s) &= \frac{1}{1-p^3X_3} F_{0,0}(X_2X_3), \\ Z_{42^+}(s) &= \frac{X_2^2X_3}{1-X_2^2X_3} F_{0,0}(X_2X_3). \end{aligned}$$

Proof. The description appearing immediately after (5.21) provides explicit formulae for $\tilde{\vartheta}(\xi_{\mathbf{e}}(\pi))$ in each of the Cases 1, 2a, 2b and 3 which, by Remark 5.9, correspond to the subcones $\mathcal{C}_{134}, \mathcal{C}_{145^+}, \mathcal{C}_{456^+}$ and $\mathcal{C}_{462^+}$ respectively. The sets $\mathcal{C}_{34}$ and $\mathcal{C}_{42^+}$ correspond to parts of Cases 1 and 3 respectively. The calculations are all similar; we show one of them. Elements of $\mathbf{e} \in \mathcal{C}_{134}$ can be expressed in the form $\mathbf{e} = r_1\mathbf{v}_1 + r_3\mathbf{v}_3 + r_4\mathbf{v}_4$, where $r_1, r_3, r_4 \in \mathbb{N}_0$, so that

$$\mathbf{e} = (e_1, e_2, e_3) = r_1\mathbf{v}_1 + r_3\mathbf{v}_3 + r_4\mathbf{v}_4 = (r_1, r_4, r_3 + r_4).$$

From this we obtain

$$\begin{aligned} Z_{134}(s) &= \sum_{\mathbf{e}\in\mathcal{C}_{134}} X_1^{e_1} X_2^{e_2} X_3^{e_3}\, \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)) \\ &= \sum_{\mathbf{e}\in\mathcal{C}_{134}} X_1^{e_1} X_2^{e_2} X_3^{e_3}\, p^{e_1-3e_2+3e_3}\, f(e_1, 0, e_2) \\ &= \sum_{r_1,r_3,r_4\geq 0} (pX_1)^{r_1}\, (p^3X_3)^{r_3}\, (X_2X_3)^{r_4}\, f(r_1, 0, r_4) \\ &= \frac{1}{1-p^3X_3} \sum_{i=0}^{\infty} (pX_1)^i\, F_{i,0}(X_2X_3). \qquad \blacksquare \end{aligned}$$

Explicit formulae for the expressions in Lemma 5.10 can now be obtained via Proposition 5.2 and Lemma 5.4. Substituting these into (5.24) yields

$$\begin{aligned} &\mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,p}(s) \\ &\quad = \frac{(1-pX_2X_3)}{(1-pX_1)(1-p^3X_3)(1-X_2^2X_3)(1-p^2X_2X_3)(1-p^3X_1^2X_2^2X_3)} \\ &\qquad \times (-p^4X_1^3X_2^3X_3^2 - p^3X_1^3X_2^2X_3 \\ &\qquad\quad -p^4X_1^2X_2^3X_3^2 - pX_1^2X_2^2X_3 + p^3X_1X_2X_3 + X_1 + pX_2X_3 + 1). \end{aligned} \tag{5.25}$$

Recalling that $X_1 = p^{14-5s}, X_2 = p^{2-s}$, $X_3 = p^{26-9s}$ we obtain

$$\begin{aligned} \zeta^{\wedge}_{\Gamma_{t^3},p}(s) &= \frac{(1-p^{29-10s})}{(1-p^{15-5s})(1-p^{29-9s})(1-p^{30-11s})(1-p^{30-10s})(1-p^{61-21s})} \\ &\quad \times (-p^{104-36s} - p^{90-31s} - p^{75-26s} - p^{59-21s} + p^{45-15s} + p^{29-10s} + p^{14-5s} + 1) \\ &= \frac{(1-p^{29-10s})}{(1-p^{15-5s})^2(1-p^{29-9s})(1-p^{30-11s})(1-p^{61-21s})} \\ &\quad \times (-p^{89-31s} - p^{75-26s} + p^{74-26s} - p^{59-21s} + p^{30-10s} - p^{15-5s} + p^{14-5s} + 1), \end{aligned}$$

proving Theorem 1.3.

6. Meromorphic continuation for the pro-isomorphic zeta function of Γ_{t^3}

In this section we consider the pro-isomorphic zeta function of the D^*-group $\Gamma = \Gamma_{t^3}$ of Hirsch length 8, defined in (1.3). Our task is to deduce the assertions about $\zeta_\Gamma^\wedge(s)$ in Corollary 1.4 from the Euler product decomposition (1.1) and the explicit description of the local zeta functions in Theorem 1.3. The main step is to establish that the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 3\}$ is a natural boundary for the meromorphic continuation of $\zeta_\Gamma^\wedge(s)$. We follow the strategy laid out in [15, Chap. 5] and use a compatible notation; in the terminology of [15], we are dealing with a Type II situation, which requires approximations up to terms of degree 3, as we shall see.

Theorem 1.3 shows that

$$\zeta_\Gamma^\wedge(s) = \frac{\zeta(5s-15)^2\zeta(9s-29)\zeta(11s-30)\zeta(21s-61)}{\zeta(10s-29)}\,\psi(s), \tag{6.1}$$

where $\zeta(s)$ denotes the Riemann zeta function and

$$\psi(s) = \prod_p \widetilde{W}(p, p^{-s}) \tag{6.2}$$

for

$$\widetilde{W}(X,Y) = 1 + X^{14}Y^5 - X^{15}Y^5 + X^{30}Y^{10} - X^{59}Y^{21} + X^{74}Y^{26} - X^{75}Y^{26} - X^{89}Y^{31},$$

as in the statement of Corollary 1.4. It is routine to check that the infinite product in (6.2) converges absolutely for all $s \in \mathbb{C}$ with

$$\mathrm{Re}(s) > \max\{15/5, 16/5, 31/10, 60/21, 75/26, 76/26, 90/31\} = 16/5$$

and yields a holomorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 16/5\}$. In passing, we observe that the abscissa of convergence of the Dirichlet generating series $\zeta_\Gamma^\wedge(s)$, which has non-negative coefficients, can be detected by looking for the rightmost singularity on the real line; from (6.1) we see that this singularity lies at $s = 30/9 = 10/3$ and yields a simple pole.

Next we show that the function $\psi(s)$, and thus $\zeta_\Gamma^\wedge(s)$, can be meromorphically continued further to the right-half plane $\mathcal{H} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 3\}$. Indeed, the cyclotomic polynomial $1 - t + t^2$ does not vanish at $t = p^{15-5s}$ for $s \in \mathcal{H}$,

because $|p^{15-5s}| < 1$. We consider

$$\begin{aligned}\widetilde{\psi}(s) &= \prod_p \frac{\widetilde{W}(p,p^{-s})}{1-p^{15-5s}+p^{30-10s}} \\ &= \prod_p \Big(1+\frac{p^{14-5s}-p^{59-21s}+p^{74-26s}-p^{75-26s}-p^{89-31s}}{1-p^{15-5s}+p^{30-10s}}\Big);\end{aligned} \tag{6.3}$$

this infinite product converges absolutely and yields a holomorphic function for $s \in \mathcal{H}$, because

$$\max\{15/5, 60/21, 75/26, 76/26, 90/31\} = 3.$$

As $1-t+t^2 = (1-t^6)(1-t)(1-t^2)^{-1}(1-t^3)^{-1}$, we see that

$$\psi(s) = \frac{\zeta(10s-30)\zeta(15s-45)}{\zeta(30s-90)\zeta(5s-15)}\,\widetilde{\psi}(s), \quad \text{for } s \in \mathcal{H},$$

yields the desired meromorphic continuation. Furthermore, using a Tauberian theorem [12, Thm. 4.20] as in the proof of Corollary 1.2, we obtain the description of the asymptotic growth of pro-isomorphic subgroups in Γ_{t^3} as recorded in Remark 1.5.

It remains to show that the line $\mathcal{L} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) = 3\}$ is a natural boundary for $\psi(s)$; in view of (6.1), this implies that $\mathcal{L}$ is also a natural boundary for $\zeta^{\wedge}_{\Gamma}(s)$ and Corollary 1.4 follows. The strategy is to show that each point $s \in \mathcal{L}$ is a limit point of zeros of the meromorphic function $\psi(s)$, defined on $\mathcal{H}$; since poles and zeros of the Riemann zeta function are isolated, it suffices to show that each $s \in \mathcal{L}$ is a limit point of zeros of the holomorphic function $\widetilde{\psi}(s)$, defined on $\mathcal{H}$. Recall from (6.3) that $\widetilde{\psi}(s)$ is given as an infinite product, indexed by p; thus $\widetilde{\psi}(s)$ vanishes, for any given $s \in \mathcal{H}$, if and only if $\widetilde{W}(p,p^{-s})$ vanishes for at least one prime p.

This leads us to study the zeros of the polynomial

$$F(V,U) = 1+(V-1)U^5+U^{10}-V^4U^{21}+(V^4-V^3)U^{26}-V^4U^{31} \in \mathbb{Z}[V][U].$$

Observe that $F(X^{-1},X^3Y) = \widetilde{W}(X,Y)$; we will be interested in evaluating F at $V = p^{-1} \to 0$, as the prime p tends to infinity, and $U = p^{3-s}$, for suitable $s \in \mathcal{H}$ depending on p. We see that

$$F(0,U) = 1-U^5+U^{10}$$

is a product of the 6th and the 30th cyclotomic polynomial. We fix the primitive 6th root of unity $\lambda = \exp(\pi i/3) = (1+\sqrt{3}\,i)/2$ so that λ is a root of $1-t+t^2$,

and we fix the primitive 30th root of unity $\omega = \exp(\pi i/15)$ so that ω is a root of $F(0,U)$. By the Holomorphic Implicit Function Theorem, there is a holomorphic function $u = u(v)$, defined in a small complex neighbourhood of $v = 0$, such that $u(0) = \omega$ and $F(v, u(v)) = 0$; furthermore, being analytic, this function admits a local representation as a power series

$$u(v) = \omega\,(1 + a_1 v + a_2 v^2 + a_3 v^3 + \cdots)$$

in v with uniquely determined complex coefficients. A routine power series calculation and comparison of coefficients yield

$$a_1 = \frac{1}{15}(2\lambda - 1), \quad a_2 = \frac{1}{15^2}(1 - 5\lambda), \quad a_3 = \frac{1}{15^3}(-17 - 450\omega + 49\omega^5 + 225\omega^6).$$

Writing $u(v) = p^{3-s}$ and $v = p^{-1}$, for sufficiently large p, we solve for $s \in \mathbb{C}$ to obtain a set

$$\mathcal{N}_p = \mathcal{H} \cap \Big\{ 3 - \underbrace{\frac{\log(\omega)}{\log(p)}}_{\in \mathbb{R}i} - \underbrace{\frac{\log(1 + a_1 p^{-1} + a_2 p^{-2} + a_3 p^{-3} + \cdots)}{\log(p)}}_{(*)} - \underbrace{\frac{2\pi k}{\log(p)} i}_{\in \mathbb{R}i} \mid k \in \mathbb{Z} \Big\}$$

of zeros of $\psi(s)$, where k is a parameter that we can use, for increasing p, to approximate any given point on the line $\mathcal{L}$ to any required degree. However, we still need to verify that, for sufficiently large p, the real part of the numerator in $(*)$ is negative so that the resulting candidate zero lies in $\mathcal{H}$, as required.

Using the logarithm series

$$\log(1 + t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots$$

for small $t = a_1 p^{-1} + a_2 p^{-2} + a_3 p^{-3} + \cdots$, we see that the relevant numerator in $(*)$ is

$$\underbrace{a_1}_{\in \mathbb{R}i} p^{-1} + \underbrace{\Big(a_2 - \frac{1}{2}a_1^2\Big)}_{\in \mathbb{R}i} p^{-2} + \underbrace{\Big(a_3 - a_1 a_2 + \frac{1}{3}a_1^3\Big)}_{\text{has negative real part}} p^{-3} + \Omega(p^{-1}), \tag{6.4}$$

where $\Omega(v)$ is a complex power series in v starting with v^4 or some higher term. Indeed, short calculations yield

$$a_1 = \frac{1}{15}(2\lambda - 1) = \frac{1}{15}\sqrt{3}\,i \in \mathbb{R}i \quad \text{and} \quad a_2 - \frac{1}{2}a_1^2 = \frac{1}{15^2}\Big(\frac{5}{2} - 5\lambda\Big) = \frac{-1}{90}\sqrt{3}\,i \in \mathbb{R}i.$$

Furthermore, a slightly longer, but routine calculation gives

$$a_3 - a_1 a_2 + \frac{1}{3}a_1^3 = \frac{1}{15^3}(-25 + 50\lambda + 15^2(\lambda - 2)\omega) = \frac{1}{135}(-1 - 18\omega + 2\omega^5 + 9\omega^6)$$

and, since $\mathrm{Re}(\omega) > \mathrm{Re}(\omega^5) > \mathrm{Re}(\omega^6)$, we deduce that

$$\mathrm{Re}\left(a_3 - a_1a_2 + \frac{1}{3}a_1^3\right) < 0$$

as asserted. For sufficiently large p, the contribution of $\Omega(p^{-1})$ in (6.4) is much smaller than the p^{-3}-term; hence $\mathcal{N}_p$ supplies the required zeros of $\psi(s)$.

7. Base extensions

Following a suggestion of the referee, we extend in this section our results for the $\mathbb{Q}$-indecomposable D^*-groups Γ_{t^2} and Γ_{t^3} to two infinite families of class-two nilpotent groups that result naturally from the initial groups via 'base extensions' of the corresponding Lie lattices; for completeness, we also discuss what happens if we start with the decomposable D^*-group Γ_t. The outcome illustrates that the investigation in [8], which was carried out partly after, partly in parallel to our original work, has an impact in the situation that we consider in this paper. We exercise some care not to exclude any primes; this allows us to get explicit results in the global setting. In a nutshell we will see that the calculations carried out in Sections 4 and 5 require only mild modifications, once the relevant algebraic automorphism groups are understood. In particular, we establish Theorem 1.6.

We briefly set up the scene. Let L be a nilpotent $\mathbb{Z}$-Lie lattice; our main interest will be in $L = L_{t^m}$, the Lie lattice associated to the nilpotent group Γ_{t^m} with presentation (1.3), with an extra focus on $m \in \{2, 3\}$. We consider a number field k of absolute degree $d = [k : \mathbb{Q}]$, with ring of integers $\mathfrak{o}$. By extension of scalars from $\mathbb{Z}$ to $\mathfrak{o}$ and restriction of scalars back to $\mathbb{Z}$, we obtain a $\mathbb{Z}$-Lie lattice $\widetilde{L} = {}_{\mathbb{Z},\mathfrak{o}}L$ of $\mathbb{Z}$-rank $\dim_{\mathbb{Z}}(\widetilde{L}) = d \dim_{\mathbb{Z}}(L)$. Clearly, $\widetilde{L}$ is nilpotent, of the same class as L.

Automorphisms of L induce in a natural way automorphisms of $\widetilde{L}$, but, in general, the automorphism group of $\widetilde{L}$ may turn out considerably more 'complex' than that of L. Consequently, the pro-isomorphic zeta functions of $\widetilde{L}$ and of L may bear little resemblance to one another. Our aim in this section is to show that Lie lattices of the form $L = L_{t^m}$, for $m \in \mathbb{N}_{\geq 2}$, are sufficiently 'rigid' so that $\mathbf{Aut}(\widetilde{L})$ is strongly linked to $\mathbf{Aut}(L)$, in an appropriate local sense. For $m \in \{2, 3\}$, this allows us to determine the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\widetilde{L},p}(s) = \zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s)$ for all primes p and, via the Euler product (1.1), we deduce analytic properties of the pro-isomorphic zeta function $\zeta^{\wedge}_{\widetilde{\Gamma}}(s)$ of the

class-two nilpotent group $\widetilde{\Gamma}$ associated to $\widetilde{L}$; compare with Section 3.1. The Lie lattice L_t is not quite 'rigid', but a slight modification of the approach in [8] allows us to bypass the problem and we obtain the local and global pro-isomorphic zeta functions also in this basic case.

7.1. Local rigidity of the Lie lattices L_{t^m} for $m \geq 2$. As above, let $\widetilde{L} = {}_{\mathbb{Z},\mathfrak{o}}L$ denote the $\mathbb{Z}$-Lie lattice associated, via 'base extension', to a $\mathbb{Z}$-Lie lattice L and a number field k with ring of integers $\mathfrak{o}$. Fix a rational prime p, and recall that there are finitely many non-archimedean primes $\mathfrak{p} \in \mathrm{Spec}(\mathfrak{o})$ dividing p. It is well known that there is a natural ring isomorphism

$$\mathbb{Z}_p\mathfrak{o} = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathfrak{o} \cong \prod_{\mathfrak{p}|p} \mathfrak{o}_{\mathfrak{p}},$$

where $\mathfrak{o}_{\mathfrak{p}}$ denotes the completion of $\mathfrak{o}$ at the prime $\mathfrak{p}$. From this one sees that the $\mathbb{Z}_p$-Lie lattice $\widetilde{L}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \widetilde{L}$, relevant to our investigation, is isomorphic to $\bigoplus_{\mathfrak{p}|p} \widetilde{L}_{\mathfrak{p}}$, where $\widetilde{L}_{\mathfrak{p}} = {}_{\mathbb{Z}_p,\mathfrak{o}_{\mathfrak{p}}}L_p$ denotes the $\mathbb{Z}_p$-Lie lattice that results from $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$ via extension of scalars to the complete valuation ring $\mathfrak{o}_{\mathfrak{p}}$ and restriction back to $\mathbb{Z}_p$.

This prompts us to consider the $\mathbb{Z}_p$-Lie lattice $\widetilde{L}_{\wp} = {}_{\mathbb{Z}_p,\mathfrak{o}}L_p$, for any given finite extension $\mathcal{F}$ of $\mathbb{Q}_p$, with valuation ring $\mathfrak{o}$ and valuation ideal $\wp$. Write $\widetilde{\mathbf{G}}_{\wp} = \mathbf{Aut}(\widetilde{L}_{\wp})$ and $\mathbf{G}_p = \mathbf{Aut}(L_p)$ for the algebraic automorphism groups of the $\mathbb{Z}_p$-Lie lattices $\widetilde{L}_{\wp}$ and L_p. Here $\mathbf{G}_p$ is simply the $\mathbb{Z}_p$-group scheme that results from the algebraic automorphism group $\mathbf{G} = \mathbf{Aut}(L)$ of the original $\mathbb{Z}$-Lie lattice via base change: any $\mathbb{Z}$-basis $\mathcal{S}$ of L naturally identifies a $\mathbb{Z}_p$-basis of L_p, and via $\mathcal{S}$ we realise $\mathbf{G}_p \leq \mathsf{GL}_n$ as an affine $\mathbb{Z}_p$-group scheme, for $n = \dim_{\mathbb{Z}}(L) = \dim_{\mathbb{Z}_p}(L_p)$. In the following we write $\mathbf{G}$ in place of $\mathbf{G}_p$, when the base ring is insignificant. Moreover, tensoring $\mathcal{S}$ with a $\mathbb{Z}_p$-basis of $\mathfrak{o}$, we obtain a $\mathbb{Z}_p$-basis $\widetilde{\mathcal{S}}$ of $\widetilde{L}_{\wp}$, which allows us to realise $\widetilde{\mathbf{G}}_{\wp} \leq \mathsf{GL}_{nd}$ as an affine $\mathbb{Z}_p$-group scheme, where $d = \dim_{\mathbb{Z}_p}(\mathfrak{o}) = [\mathcal{F} : \mathbb{Q}_p]$. Our explicit construction yields, in particular,

$$\mathbf{G}(\mathfrak{o}) \cong \mathrm{Aut}({}_{\mathfrak{o}}L_p) \leq \mathrm{Aut}({}_{\mathbb{Z}_p,\mathfrak{o}}L_p) \cong \widetilde{\mathbf{G}}_{\wp}(\mathbb{Z}_p),$$
$$\mathbf{G}(\mathcal{F}) \cong \mathrm{Aut}({}_{\mathcal{F}}L_p) \leq \mathrm{Aut}({}_{\mathbb{Q}_p,\mathcal{F}}L_p) \cong \widetilde{\mathbf{G}}_{\wp}(\mathbb{Q}_p).$$

Typically, these embeddings are proper, because $\mathbb{Z}_p$-linear automorphisms are not necessarily $\mathfrak{o}$-linear. Suppose that L, hence also L_p, is nilpotent of class 2. In this situation we can easily make out two types of automorphisms, which

could be used to fill this gap: central automorphisms and field automorphisms. More precisely, we set

$$\mathbf{J}_\wp = \mathsf{C}_{\widetilde{\mathbf{G}}_\wp}(\widetilde{L}_\wp/\mathrm{Z}(\widetilde{L}_\wp)) = \ker(\mathbf{Aut}(\widetilde{L}_\wp) \to \mathbf{Aut}(\widetilde{L}_\wp/\mathrm{Z}(\widetilde{L}_\wp))) \trianglelefteq \widetilde{\mathbf{G}}_\wp,$$

the affine $\mathbb{Z}_p$-group scheme which is the algebraic centraliser of the $\mathbb{Z}_p$-module $\widetilde{L}_\wp/\mathrm{Z}(\widetilde{L}_\wp)$. For the concrete realisation as a subgroup scheme in GL_{nd}, it is convenient to choose the underlying $\mathbb{Z}_p$-basis $\mathcal{S}$ in such a way that it includes a $\mathbb{Z}_p$-basis for $\mathrm{Z}(L_p)$; then $\widetilde{\mathcal{S}}$ includes a $\mathbb{Z}_p$-basis for $\mathrm{Z}(\widetilde{L}_\wp)$ and $\mathbf{J}_\wp$ can be defined rather directly. In addition, we consider the algebraic automorphism group of the extension $\mathcal{F} \,|\, \mathbb{Q}_p$ as a subgroup scheme of $\mathbf{Aut}(\widetilde{L}_\wp)$, that is the finite group scheme

$$\mathbf{F}_\wp \leq \widetilde{\mathbf{G}}_\wp \quad \text{with } \mathbf{F}_\wp \cong \mathbf{Aut}(\mathfrak{o} \,|\, \mathbb{Z}_p) \cong \mathbf{Aut}(\mathcal{F} \,|\, \mathbb{Q}_p)$$

such that, in particular, $\mathbf{F}_\wp(\mathbb{Z}_p) \cong \mathrm{Aut}(\mathfrak{o} \,|\, \mathbb{Z}_p) \cong \mathrm{Aut}(\mathcal{F} \,|\, \mathbb{Q}_p)$ acts naturally via field automorphisms on the Lie lattice $\widetilde{L}_\wp$. Furthermore, we observe that $\mathbf{G}$, now regarded as an affine $\mathfrak{o}$-group scheme, serves as the algebraic automorphism group $\mathbf{Aut}({}_\mathfrak{o}L_p)$ of the $\mathfrak{o}$-Lie lattice ${}_\mathfrak{o}L_p$; accordingly, the affine $\mathbb{Z}_p$-group scheme $\mathsf{Res}_{\mathfrak{o} \,|\, \mathbb{Z}_p}(\mathbf{G})$ which results via restriction of scalars can be realised as a subgroup scheme of $\widetilde{\mathbf{G}}_\wp$. We are interested in situations where the following 'rigidity' holds:

$$(7.1)\quad (\mathbf{J}_\wp \cdot \mathsf{Res}_{\mathfrak{o}|\mathbb{Z}_p}(\mathbf{G})) \rtimes \mathbf{F}_\wp = \widetilde{\mathbf{G}}_\wp \quad \text{as } \mathbb{Q}_p\text{-defined algebraic subgroups of } \mathsf{GL}_{nd}.$$

Actually, for us it suffices that the two group schemes yield the same groups of $\mathbb{Q}_p$-rational points; this condition is slightly weaker, but implies, for instance, that the two $\mathbb{Q}_p$-algebraic groups have the same connected component. In down-to-earth terms we require that the $\mathcal{F}$- and thus also $\mathbb{Q}_p$-Lie algebra $\widetilde{\mathcal{L}} = {}_\mathcal{F}L_p = \mathcal{F} \otimes_{\mathbb{Z}_p} L_p$ satisfies

$$(7.2)\qquad (\mathrm{C}_{\mathrm{Aut}_{\mathbb{Q}_p}(\widetilde{\mathcal{L}})}(\widetilde{\mathcal{L}}/\mathrm{Z}(\widetilde{\mathcal{L}}))\ \mathrm{Aut}_\mathcal{F}(\widetilde{\mathcal{L}})) \rtimes \mathrm{Aut}(\mathcal{F} \,|\, \mathbb{Q}_p) = \mathrm{Aut}_{\mathbb{Q}_p}(\widetilde{\mathcal{L}}).$$

In [8], Berman, Glazer and Schein extend results of Segal [30] for algebraic automorphism groups of certain Lie algebras, with a view toward studying pro-isomorphic zeta functions under 'base extensions'. In particular, they formulate sufficient conditions under which (7.1) holds true; see [8, Thm. 3.9]. For the discussion at hand, a special and thus simpler version of their criterion is sufficient. We say that the $\mathbb{Q}_p$-Lie algebra ${}_{\mathbb{Q}_p}L \cong {}_{\mathbb{Q}_p}L_p$ is **absolutely indecomposable** if, for every finite extension $\mathcal{F}$ of $\mathbb{Q}_p$, the $\mathcal{F}$-Lie algebra ${}_\mathcal{F}L \cong {}_\mathcal{F}L_p$ is indecomposable. We make use of the following special instance of [8, Thm. 3.9].

LEMMA 7.1: *Let L be a class-two nilpotent $\mathbb{Z}$-Lie lattice, as above, and such that $[L,L] = \mathrm{Z}(L)$. Let p be a prime such that the $\mathbb{Q}_p$-Lie algebra $\mathcal{L} = {}_{\mathbb{Q}_p}L_p$ is absolutely indecomposable and generated by*

$$\mathcal{Y} = \{w \in \mathcal{L} \smallsetminus \mathrm{Z}(\mathcal{L}) \mid \mathrm{C}_{\mathcal{L}}(\mathrm{C}_{\mathcal{L}}(w)) = \mathbb{Q}_p w + \mathrm{Z}(\mathcal{L})\}.$$

Then (7.1) holds, for every finite extension $\mathcal{F}$ of $\mathbb{Q}_p$, with valuation ring $\mathfrak{o}$ and valuation ideal $\wp$.

Next we consider the $\mathbb{Z}$-Lie lattices L_{t^m}, $m \in \mathbb{N}$, associated to the D^*-groups Γ_{t^m} with presentation (1.3). This means that L_{t^m} has $\mathbb{Z}$-rank $2m+2$ and admits the presentation

$$\begin{aligned} L_{t^m} = \langle x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, z_2 \mid & \\ [x_i, y_j] = \delta_{i,j} z_1 + \delta_{i+1,j} z_2, & \\ [x_i, x_j] = [y_i, y_j] = [x_i, z_1] = [x_i, z_2] = [y_i, z_1] = [y_i, z_2] = 0 & \\ \text{for } 1 \leq i, j \leq m \rangle, & \end{aligned} \tag{7.3}$$

a special instance of (2.2). Furthermore, $\mathrm{Z}(L) = \mathbb{Z}z_1 + \mathbb{Z}z_2$, and $[L,L] = \mathrm{Z}(L)$ for $m \geq 2$.

LEMMA 7.2: *Let $L = L_{t^m}$ with $m \geq 2$, and let $\mathcal{F}$ be any field. Then the $\mathcal{F}$-Lie algebra $\mathcal{L} = {}_{\mathcal{F}}L$ is indecomposable.*

Proof. Put $\mathcal{Z} = \mathrm{Z}(\mathcal{L}) = \mathrm{span}_{\mathcal{F}}\{z_1, z_2\}$. A routine check shows that

$$\begin{aligned} \mathcal{W} &= \{w \in \mathcal{L} \mid \dim_{\mathcal{F}}(\mathrm{span}_{\mathcal{F}}\{[w,v] \mid v \in \mathcal{L}\}) \leq 1\} \\ &= \{w \in \mathcal{L} \mid \mathrm{span}_{\mathcal{F}}\{[w,v] \mid v \in \mathcal{L}\} \subseteq \mathcal{F}z_1\} = \mathrm{span}_{\mathcal{F}}\{x_m, y_1\} + \mathcal{Z} \end{aligned} \tag{7.4}$$

so that $\mathcal{W}$ is a vector subspace and $\dim_{\mathcal{F}}(\mathcal{W}) = 4$. For a contradiction, suppose that $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$ for non-zero Lie ideals $\mathcal{A}, \mathcal{B} \trianglelefteq \mathcal{L}$. Since $\mathcal{A}$ is nilpotent, $\mathcal{A}$ has non-zero centre $\mathrm{Z}(\mathcal{A}) \neq \{0\}$, and likewise $\mathrm{Z}(\mathcal{B}) \neq \{0\}$. Thus $\mathcal{Z} = \mathrm{Z}(\mathcal{A}) \oplus \mathrm{Z}(\mathcal{B})$ implies $\dim_{\mathcal{F}}(\mathrm{Z}(\mathcal{A})) = \dim_{\mathcal{F}}(\mathrm{Z}(\mathcal{B})) = 1$. We deduce that $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{W}$ and hence $\mathcal{W} = \mathcal{L}$, in contradiction to $\dim_{\mathcal{F}}(\mathcal{L}) = 2m + 2 > 4$. ∎

We remark that, in contrast to the situation treated in Lemma 7.2, the Lie lattice L_t is already decomposable over $\mathbb{Z}$: clearly, $L_t = (\mathbb{Z}x_1 + \mathbb{Z}y_1 + \mathbb{Z}z_1) \oplus \mathbb{Z}z_2$ decomposes as a direct sum of two non-zero Lie ideals.

LEMMA 7.3: *Let $L = L_{t^m}$ with $m \geq 2$, and let $\mathcal{F}$ be any field. Then the $\mathcal{F}$-Lie algebra $\mathcal{L} = {}_{\mathcal{F}}L$ is generated by*

$$\mathcal{Y} = \{w \in \mathcal{L} \smallsetminus \mathrm{Z}(\mathcal{L}) \mid \mathrm{C}_{\mathcal{L}}(\mathrm{C}_{\mathcal{L}}(w)) = \mathcal{F}w + \mathrm{Z}(\mathcal{L})\}$$

if and only if $m \neq 2$.

Proof. For short we put $\mathcal{Z} = \mathrm{Z}(\mathcal{L}) = [\mathcal{L}, \mathcal{L}] = \mathrm{span}_{\mathcal{F}}\{z_1, z_2\}$.

First consider the special case $m = 2$. We claim that $\mathcal{Y}$ is contained in the proper Lie subalgebra $\mathcal{W} = \mathrm{span}_{\mathcal{F}}\{x_2, y_1\} + \mathcal{Z}$; thus $\mathcal{Y}$ fails to generate $\mathcal{L}$. Indeed, from the description (7.4) and the definition of $\mathcal{Y}$ we see that both $\mathcal{W}$ and $\mathcal{Y}$ are $\mathrm{Aut}(\mathcal{L})$-invariant. Thus it suffices to check that $x_1 \notin \mathcal{Y}$ and that for every $w \in \mathcal{L} \smallsetminus \mathcal{W}$ there exists $g \in \mathrm{Aut}(\mathcal{L})$ such that $wg = x_1$. From $\mathrm{C}_{\mathcal{L}}(x_1) = \mathrm{span}_{\mathcal{F}}\{x_1, x_2\} + \mathcal{Z}$ we deduce that $\mathrm{C}_{\mathcal{L}}(\mathrm{C}_{\mathcal{L}}(x_1)) = \mathrm{span}_{\mathcal{F}}\{x_1, x_2\} + \mathcal{Z}$, and this gives $x_1 \notin \mathcal{Y}$. Now let $w \in \mathcal{L} \smallsetminus \mathcal{W}$. Corollary 2.7 describes the reductive part of $\mathrm{Aut}(\mathcal{L})$; compare with (4.1). From this description we see that there exists $g_1 \in \mathrm{Aut}(\mathcal{L})$ such that $wg_1 \in x_1 + \mathcal{W}$. Finally, the description of the unipotent radical of $\mathrm{Aut}(\mathcal{L})$ in Example 2.9 shows that there exists $g_2 \in \mathrm{Aut}(\mathcal{L})$ such that $wg_1g_2 = x_1$.

Now suppose that $m \geq 3$. We claim that $\mathcal{Y}$ contains the generating set

$$x_1,\, x_2,\, \ldots,\, x_{m-2},\, x_m,\, y_1,\, y_3,\, y_4,\, \ldots,\, y_m,\, \sum_{i=1}^{m} x_i,\, \sum_{i=1}^{m} y_i$$

for $\mathcal{L}$. Indeed, for $i \in \{1, \ldots, m\}$ it is easily checked that

$$\mathrm{C}_{\mathcal{L}}(x_i) = \mathrm{span}_{\mathcal{F}}\{x_1, \ldots, x_m, y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_m\} + \mathcal{Z},$$
$$\mathrm{C}_{\mathcal{L}}(y_i) = \mathrm{span}_{\mathcal{F}}\{x_1, \ldots, x_{i-2}, x_{i+1}, \ldots, x_m, y_1, \ldots, y_m\} + \mathcal{Z}.$$

For $i \neq m-1$ this implies $\mathrm{C}_{\mathcal{L}}(\mathrm{C}_{\mathcal{L}}(x_i)) = \mathcal{F}x_i + \mathcal{Z}$, hence $x_i \in \mathcal{Y}$. Likewise $y_i \in \mathcal{Y}$ for $i \neq 2$, but it can be seen that x_{m-1}, y_2 do not belong to $\mathcal{Y}$. In order to bypass these exceptions, it suffices to show that $\sum_{i=1}^{m} x_i$ and $\sum_{i=1}^{m} y_i$ lie in $\mathcal{Y}$. We deduce from

$$\mathrm{C}_{\mathcal{L}}\Big(\sum_{i=1}^{m} x_i\Big) = \mathrm{span}_{\mathcal{F}}\{x_1, \ldots, x_m, y_2 - y_3, y_3 - y_4, \ldots, y_{m-1} - y_m\} + \mathcal{Z}$$

that

$$\mathrm{C}_{\mathcal{L}}\Big(\mathrm{C}_{\mathcal{L}}\Big(\sum_{i=1}^{m} x_i\Big)\Big) = \mathcal{F}\Big(\sum_{i=1}^{m} x_i\Big) + \mathcal{Z}.$$

This gives $\sum_{i=1}^{m} x_i \in \mathcal{Y}$ and similarly $\sum_{i=1}^{m} y_i \in \mathcal{Y}$. ∎

We remark that, for $m = 1$, the set $\mathcal{Y} \subseteq \mathcal{L} = {}_{\mathcal{F}}L_t$ defined in Lemma 7.3 coincides with $\mathcal{L} \smallsetminus \mathrm{Z}(\mathcal{L})$ and thus generates $\mathcal{L}$ for trivial reasons.

PROPOSITION 7.4: *Let $L = L_{t^m}$ with $m \geq 2$, and let p be a prime. Then* (7.2) *holds for every finite extension $\mathcal{F}$ of $\mathbb{Q}_p$.*

Proof. For $m > 2$ we can use the criterion established in [8, Thm. 3.9]: the stronger 'rigidity condition' (7.1) follows, for every finite extension $\mathcal{F}$ of $\mathbb{Q}_p$, with valuation ring $\mathfrak{o}$ and valuation ideal $\wp$, from Lemmata 7.1, 7.2 and 7.3. For $m = 2$ we give a direct proof of (7.2), as follows.

Fix a finite extension $\mathcal{F}$ of $\mathbb{Q}_p$ of degree $d = [\mathcal{F} : \mathbb{Q}_p]$ and pick a primitive element α for the extension so that

$$\mathcal{F} = \mathbb{Q}_p(\alpha) = \mathbb{Q}_p\, 1 + \mathbb{Q}_p\, \alpha + \cdots + \mathbb{Q}_p\, \alpha^{d-1}.$$

The $\mathbb{Q}_p$-Lie algebra $\widetilde{\mathcal{L}} = {}_{\mathbb{Q}_p,\mathcal{F}}\mathcal{L}$ results from the 6-dimensional $\mathbb{Q}_p$-Lie algebra $\mathcal{L} = \mathbb{Q}_p \otimes_{\mathbb{Z}} L$ with basis $x_1, x_2, y_1, y_2, z_1, z_2$, subject to the relations indicated in (7.3), via extension and restriction of scalars; we have $\dim_{\mathbb{Q}_p}(\widetilde{\mathcal{L}}) = 6d$ and $\widetilde{\mathcal{L}}$ admits a $\mathbb{Q}_p$-basis consisting of the elementary tensors

$$x_i\alpha^j = \alpha^j \otimes x_i, \quad y_i\alpha^j = \alpha^j \otimes y_i, \quad z_i\alpha^j = \alpha^j \otimes z_i, \quad \text{for } i \in \{1,2\},\ j \in \{0,\ldots,d-1\},$$

where we write the powers of α on the right so that they are visibly separated from scalars coming from $\mathbb{Q}_p$. Likewise we find it convenient in the calculations below to treat $\widetilde{\mathcal{L}}$ formally as a $(\mathbb{Q}_p, \mathcal{F})$-bimodule. We put $\widetilde{\mathcal{Z}} = \mathrm{Z}(\widetilde{\mathcal{L}})$ and recall that

$$[\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}] = \widetilde{\mathcal{Z}} = \mathrm{span}_{\mathcal{F}}\{z_1, z_2\} = \mathrm{span}_{\mathbb{Q}_p}\{z_i\alpha^j \mid i \in \{1,2\},\ j \in \{0,\ldots,d-1\}\}.$$

Furthermore, we observe that with

$$\begin{aligned} \widetilde{\mathcal{W}} &= \mathrm{span}_{\mathcal{F}}\{x_2, y_1\} + \mathrm{Z}(\widetilde{\mathcal{L}}) = \{w \in \widetilde{\mathcal{L}} \mid \dim_{\mathcal{F}}[w, \widetilde{\mathcal{L}}] \leq 1\} \\ &= \mathrm{span}_{\mathbb{Q}_p}\{x_2\alpha^j \mid 0 \leq j < d\} \cup \{y_1\alpha^j \mid 0 \leq j < d\} + \mathrm{Z}(\widetilde{\mathcal{L}}) \\ &= \{w \in \widetilde{\mathcal{L}} \mid \dim_{\mathbb{Q}_p}[w, \widetilde{\mathcal{L}}] \leq d\} \end{aligned} \tag{7.5}$$

we obtain a chain of $\mathrm{Aut}_{\mathbb{Q}_p}(\widetilde{\mathcal{L}})$-invariant $\mathcal{F}$- and hence $\mathbb{Q}_p$-subspaces

$$\{0\} \subseteq \underbrace{[\widetilde{\mathcal{W}}, \widetilde{\mathcal{L}}]}_{=z_1\mathcal{F}} \subseteq \widetilde{\mathcal{Z}} \ \subseteq\ \widetilde{\mathcal{W}} \subseteq \widetilde{\mathcal{L}} \tag{7.6}$$

with $\dim_{\mathbb{Q}_p}[\widetilde{\mathcal{W}}, \widetilde{\mathcal{L}}] = d$ and $\dim_{\mathbb{Q}_p} \widetilde{\mathcal{W}} = 4d$; compare with (7.4).

Now consider an arbitrary automorphism $\varphi \in \mathrm{Aut}_{\mathbb{Q}_p}(\widetilde{\mathcal{L}})$. By means of a finite number of basic reductions, we show that φ is contained in the subgroup that appears on the left-hand side of (7.2).

Step 1. By Proposition 2.4, the group $\mathrm{Aut}_{\mathcal{F}}(\widetilde{L})$ induces on $\widetilde{\mathcal{Z}} = z_2\mathcal{F} + z_1\mathcal{F}$ the group of all invertible upper triangular matrices; in particular, it acts transitively on $(z_1\mathcal{F} \smallsetminus \{0\}) \times ((z_2\mathcal{F} + z_1\mathcal{F}) \smallsetminus z_1\mathcal{F})$. In view of the φ-invariance of $z_1\mathcal{F}$ and $z_2\mathcal{F} + z_1\mathcal{F}$ in (7.6) we may thus suppose without loss of generality that φ fixes z_1 and z_2:

$$z_1\varphi = z_1 \quad \text{and} \quad z_2\varphi = z_2.$$

Step 2. Next we focus on $[\widetilde{W}, \widetilde{L}] = z_1\mathcal{F} = \mathrm{span}_{\mathbb{Q}_p}\{z_1\alpha^j \mid 0 \le j < d\}$, with the aim to reduce to the situation where φ induces the identity on this subspace. In view of (7.6) we may write

$$(z_1\alpha^j)\varphi = z_1\lambda_j \quad \text{for suitable } \lambda_j \in \mathcal{F},\ 0 \le j \le d.$$

Due to the reduction in Step 1 we have $\lambda_0 = 1$, and λ_d is actually determined by $\lambda_0, \dots, \lambda_{d-1}$, because α^d can be expressed as a $\mathbb{Q}_p$-linear combination of $\alpha^0, \dots, \alpha^{d-1}$; in (7.7) below it becomes clear why our analysis includes λ_d. Furthermore, for $0 \le j < d$, the images of $x_1\alpha^j$ and $y_1\alpha^j \in \widetilde{\mathcal{W}}$ under φ can be written, modulo $\widetilde{\mathcal{Z}}$, as $\mathcal{F}$-linear combinations

$$(x_1\alpha^j)\varphi \equiv_{\widetilde{\mathcal{Z}}} x_1a_j + y_2b_j + x_2a'_j + y_1b'_j \quad \text{and} \quad (y_1\alpha^j)\varphi \equiv_{\widetilde{\mathcal{Z}}} x_2c_j + y_1d_j.$$

For $0 \le j < d$ we deduce that

$$0 = [x_1, x_1\alpha^j]\varphi = [x_1a_0 + y_2b_0 + \cdots, x_1a_j + y_2b_j + \cdots] \equiv_{z_1\mathcal{F}} z_2(a_0b_j - b_0a_j)$$

so that $a_0b_j = a_jb_0$. In a similar way, for $0 \le j \le d$ and $0 \le i \le \min\{1, j\}$ we see that

$$\begin{aligned} z_1\lambda_j &= (z_1\alpha^j)\varphi = [x_1\alpha^{j-i}, y_1\alpha^i]\varphi \\ &= [x_1a_{j-i} + y_2b_{j-i} + \dots, x_2c_i + y_1d_i] = z_1(a_{j-i}d_i - b_{j-i}c_i) \end{aligned}$$

so that $\lambda_j = a_{j-i}d_i - b_{j-i}c_i$. Using $b_0a_{j-1} = a_0b_{j-1}$ to modify the underlined terms and $\lambda_0 = 1$ for the final simplification, we deduce that for $1 \le j \le d$,

$$\begin{aligned} \lambda_1\lambda_{j-1} &= (a_0d_1 - b_0c_1)(a_{j-1}d_0 - b_{j-1}c_0) \\ &= a_0d_1a_{j-1}d_0 - \underline{b_0c_1a_{j-1}d_0} - \underline{a_0d_1b_{j-1}c_0} + b_0c_1b_{j-1}c_0 \\ &= a_0d_0(a_{j-1}d_1 - b_{j-1}c_1) - b_0c_0(a_{j-1}d_1 - b_{j-1}c_1) \\ &= (a_0d_0 - b_0c_0)\lambda_j = \lambda_0\lambda_j = \lambda_j. \end{aligned}$$

By induction, we obtain $\lambda_j = \lambda_1^j$ for $0 \leq j \leq d$. Let $f = \sum_{j=0}^{d} f_j t^j \in \mathbb{Q}_p[t]$ denote the minimal polynomial of α over $\mathbb{Q}_p$. Then

$$(7.7)\quad 0 = (z_1 f(\alpha))\varphi = \sum_{j=0}^{d} f_j (z_1 \alpha^j)\varphi = \sum_{j=0}^{d} f_j (z_1 \lambda_j) = z_1 \Big(\sum_{j=0}^{d} f_j \lambda_j \Big) = z_1 f(\lambda_1)$$

implies $f(\lambda_1) = 0$. Hence α and λ_1 are Galois conjugates in $\mathcal{F} = \mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\lambda_1)$. Modifying φ by a field automorphism, i.e., an element of $\mathrm{Aut}(\mathcal{F} \,|\, \mathbb{Q}_p)$, we may suppose without loss of generality that

$$(z_1 \alpha^j)\varphi = z_1 \alpha^j \quad \text{for } 0 \leq j < d.$$

Step 3. Next we focus on the action of φ on $\widetilde{\mathcal{Z}} = z_1\mathcal{F} + z_2\mathcal{F}$ modulo $[\widetilde{W}, \widetilde{L}] = z_1\mathcal{F}$; this factor space admits $z_2\alpha^j$, $0 \leq j < d$, as a $\mathbb{Q}_p$-basis. In view of (7.6) we may write

$$(z_2 \alpha^j)\varphi \equiv_{z_1\mathcal{F}} z_2 \mu_j \quad \text{for suitable } \mu_j \in \mathcal{F},\ 0 \leq j \leq d;$$

our aim is to show that $\mu_j = \beta^j$, with $\beta = \mu_1$ Galois conjugate to α.

Due to the reduction in Step 1 we have $\mu_0 = 1$, and μ_d is actually determined by $\mu_0, \ldots, \mu_{d-1}$; compare with Step 2. For $0 \leq j < d$, the images of $x_1\alpha^j$ and $y_2\alpha^j$ under φ can be written, modulo $\widetilde{\mathcal{Z}}$, as $\mathcal{F}$-linear combinations

$$(x_1\alpha^j)\varphi \equiv_{\widetilde{\mathcal{Z}}} x_1 a_j + y_2 b_j + x_2 a'_j + y_1 b'_j \quad \text{and} \quad (y_2\alpha^j)\varphi \equiv_{\widetilde{\mathcal{Z}}} x_1 c_j + y_2 d_j + x_2 c'_j + y_1 d'_j.$$

In Step 2 we saw that $a_0 b_j = a_j b_0$ for $0 \leq j < d$. Furthermore, for $0 \leq j \leq d$ and $0 \leq i \leq \min\{1, j\}$ we get, modulo $z_1\mathcal{F}$,

$$\begin{aligned} z_2\mu_j &\equiv_{z_1\mathcal{F}} (z_2\alpha^j)\varphi = [x_1\alpha^{j-i}, y_2\alpha^i]\varphi \\ &= [x_1 a_{j-i} + y_2 b_{j-i} + \cdots, x_1 c_i + y_2 d_i + \cdots] \equiv_{z_1\mathcal{F}} z_2(a_{j-i}d_i - b_{j-i}c_i) \end{aligned}$$

so that $\mu_j = a_{j-i}d_i - b_{j-i}c_i$. A similar argument as in Step 2 shows that $\mu_j = \mu_1^j$ for $0 \leq j \leq d$ and that α and $\beta = \mu_1$ are Galois conjugates in $\mathcal{F} = \mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\beta)$.

Step 4. We analyse further the action of φ on $\widetilde{\mathcal{Z}} = z_1\mathcal{F} + z_2\mathcal{F}$. So far we have reduced to the situation in which φ acts as the identity on $z_1\mathcal{F}$ and

$$(z_2\alpha^j)\varphi = z_2\beta^j + z_1\nu_j \quad \text{for } 1 \leq j \leq d,$$

where β denotes a Galois conjugate of α and $0 = \nu_0, \nu_1, \ldots, \nu_d \in \mathcal{F}$ are suitable coefficients. As before, ν_d is actually determined by the previous parameters. Proposition 2.6 describes the pointwise stabiliser of $\mathrm{Z}(\widetilde{\mathcal{L}})$ inside $\mathrm{Aut}_{\mathcal{F}}(\widetilde{\mathcal{L}})$; a short

reflection reveals that this stabiliser acts transitively on $\widetilde{\mathcal{L}} \smallsetminus \widetilde{\mathcal{W}}$ and consequently we may suppose without loss of generality that

$$x_1\varphi = x_1;$$

in particular, the abelian Lie subalgebra

$$\widetilde{\mathfrak{X}} = \mathrm{C}_{\widetilde{\mathcal{L}}}(x_1) = \mathrm{C}_{\widetilde{\mathcal{L}}}(x_1\mathcal{F} + x_2\mathcal{F}) = \mathrm{span}_{\mathcal{F}}\{x_1, x_2, z_1, z_2\}$$

is φ-invariant. For $0 \leq j \leq d$ we deduce from

$$[x_1, (y_2\alpha^j)\varphi] = [x_1, y_2\alpha^j]\varphi = (z_2\alpha^j)\varphi = z_2\beta^j + z_1\nu_j$$

that, modulo $\widetilde{\mathfrak{X}}$,

$$(y_2\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{X}}} y_1\nu_j + y_2\beta^j;$$

in particular, $y_2\varphi \equiv y_2$ modulo $\widetilde{\mathfrak{X}}$. Furthermore, $(x_1\alpha^j)\varphi \in \widetilde{\mathfrak{X}}$ and

$$[(x_1\alpha^j)\varphi, y_2] = [(x_1\alpha^j)\varphi, y_2\varphi] = [x_1\alpha^j, y_2]\varphi = (z_2\alpha^j)\varphi = z_2\beta^j + z_1\nu_j$$

yield, modulo $\widetilde{\mathcal{Z}}$,

$$(x_1\alpha^j)\varphi \equiv_{\widetilde{\mathcal{Z}}} x_1\beta^j + x_2\nu_j.$$

For $0 \leq j < d$ we deduce from

$$\begin{aligned} z_2\beta^{j+1} + z_1(\beta^j\nu_1 + \beta\nu_j) &= [x_1\beta^j + x_2\nu_j, y_1\nu_1 + y_2\beta] = [x_1\alpha^j, y_2\alpha]\varphi \\ &= (z_2\alpha^{j+1})\varphi = z_2\beta^{j+1} + z_1\nu_{j+1} \end{aligned}$$

that $\nu_{j+1} = \beta\nu_j + \beta^j\nu_1$. By recursion, this gives

$$\nu_j = j\beta^{j-1}\nu_1 \quad \text{for } 0 \leq j \leq d.$$

Let $f = \sum_{j=0}^{d} f_j t^j \in \mathbb{Q}_p[t]$ denote the minimal polynomial of α and of its conjugate β over $\mathbb{Q}_p$. Then

$$0 = (z_2 \underbrace{f(\alpha)}_{=0})\varphi = \left(\sum_{j=0}^{d} f_j(z_2\alpha^j)\right)\varphi = \sum_{j=0}^{d} f_j((z_2\alpha^j)\varphi) = \sum_{j=0}^{d} f_j\,(z_2\beta^j + z_1\nu_j)$$

$$= \sum_{j=0}^{d} f_j\,(z_2\beta^j + z_1(j\beta^{j-1}\nu_1)) = z_2\underbrace{f(\beta)}_{=0} + z_1(\nu_1 f'(\beta)) = z_1(\nu_1 \underbrace{f'(\beta)}_{\neq 0})$$

implies $\nu_1 = 0$, hence $\nu_j = 0$ for $0 \leq j \leq d$ and

$$(z_2\alpha^j)\varphi = z_2\beta^j \quad \text{for } 0 \leq j < d.$$

STEP 5. Finally, let us see how φ acts modulo the centre. In Step 4 we saw that $y_2\varphi \equiv y_2$ modulo $\widetilde{\mathfrak{X}}$. Proposition 2.6 describes the pointwise stabiliser of $\widetilde{\mathfrak{Z}}$ inside $\mathrm{Aut}_{\mathcal{F}}(\widetilde{\mathcal{L}})$; in particular, this stabiliser acts transitively on $y_2 + \widetilde{\mathfrak{X}}$, even if we add the condition that x_1 is to remain fixed: in the notation of the proposition, we can take

$$X_1 = \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \text{where } c_1, c_2 \in \mathcal{F} \text{ are free parameters.}$$

Thus we may suppose, without interfering with the previous reductions, that φ fixes y_2, i.e.,

$$y_2\varphi = y_2.$$

From $x_2\varphi \in \widetilde{\mathfrak{X}}$ and $[x_2\varphi, y_2] = [x_2, y_2]\varphi = z_1\varphi = z_1$ we deduce that $x_2\varphi \equiv x_2$ modulo $\widetilde{\mathfrak{Z}}$. Recall that $y_1 \in \widetilde{\mathcal{W}}$ implies $y_1\varphi \in \widetilde{\mathcal{W}}$; moreover, φ fixes x_1 and z_1. Hence $[x_1, y_1\varphi] = [x_1, y_1]\varphi = z_1\varphi = z_1$ gives $y_1\varphi \equiv y_1$ modulo $x_2\mathcal{F} + \widetilde{\mathfrak{Z}}$. From $[y_1\varphi, y_2] = [y_1, y_2]\varphi = 0$ we conclude that $y_1\varphi \equiv y_1$ modulo $\widetilde{\mathfrak{Z}}$. We have gained

$$x_2\varphi \equiv_{\widetilde{\mathfrak{Z}}} x_2 \quad \text{and} \quad y_1\varphi \equiv_{\widetilde{\mathfrak{Z}}} y_1.$$

Now let $0 \leq j < d$. From

$$[x_1, (y_1\alpha^j)\varphi] = [x_1, y_1\alpha^j]\varphi = (z_1\alpha^j)\varphi = z_1\alpha^j,$$
$$[y_1, (y_1\alpha^j)\varphi] = [y_2, (y_1\alpha^j)\varphi] = 0$$

we see that $(y_1\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{Z}}} y_1\alpha^j$. Similarly, $[x_1, (y_2\alpha^j)\varphi] = (z_2\alpha^j)\varphi = z_2\beta^j$ and $[y_1, (y_2\alpha^j)\varphi] = [y_2, (y_2\alpha^j)\varphi] = 0$ imply $(y_2\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{Z}}} y_2\beta^j$. Moreover

$$z_1\alpha = (z_1\alpha)\varphi = [x_2, y_2\alpha]\varphi = [x_2, (y_2\alpha)\varphi] = [x_2, y_2\beta] = z_1\beta$$

implies $\alpha = \beta$.

In summary, this shows that φ fixes pointwise the centre $\widetilde{\mathfrak{Z}}$, and that, modulo $\widetilde{\mathfrak{Z}}$,

$$(y_1\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{Z}}} y_1\alpha^j \quad \text{and} \quad (y_2\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{Z}}} y_2\alpha^j, \quad \text{for } 0 \leq j < d.$$

Finally, we observe that $(x_1\alpha^j)\varphi, (x_2\alpha^j)\varphi \in \widetilde{\mathfrak{X}}$ satisfy

$$[(x_1\alpha^j)\varphi, y_2] = [x_1\alpha^j, y_2]\varphi = (z_2\alpha^j)\varphi = z_2\alpha^j$$

and, by similar considerations, $[(x_2\alpha^j)\varphi, y_1] = 0$ and $[(x_2\alpha^j)\varphi, y_2]\varphi = z_1\alpha^j$. From this we conclude that, modulo $\widetilde{\mathfrak{Z}}$,

$$(x_1\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{Z}}} x_1\alpha^j \quad \text{and} \quad (x_2\alpha^j)\varphi \equiv_{\widetilde{\mathfrak{Z}}} x_2\alpha^j, \quad \text{for } 0 \leq j < d.$$

As $\widetilde{\mathfrak{Z}} = \mathrm{Z}(\widetilde{\mathcal{L}})$ it follows that $\varphi \in \mathrm{C}_{\mathrm{Aut}_{\mathbb{Q}_p}(\widetilde{\mathcal{L}})}(\widetilde{\mathcal{L}}/\mathrm{Z}(\widetilde{\mathcal{L}}))$ is contained in the subgroup on the left-hand side of (7.2). ∎

7.2. The local pro-isomorphic zeta functions of groups $\widetilde{\Gamma}$ associated to L_{t^2} and L_{t^3}. We return to the setting described at the beginning of the section. Let k be a number field of absolute degree $d = [k : \mathbb{Q}]$, with ring of integers $\mathfrak{o}$. Let $\widetilde{L} = {}_{\mathbb{Z},\mathfrak{o}}L$ be the nilpotent $\mathbb{Z}$-Lie lattice associated to L_{t^m} for $m \in \{2, 3\}$ via 'base extension', with algebraic automorphism group $\widetilde{\mathbf{G}} = \mathbf{Aut}(\widetilde{L}) \leq \mathsf{GL}_{nd}$, where $n = \dim_{\mathbb{Z}} L = 2m+2$, and let p be a prime. The basic ingredients for the 'fine' Euler decomposition established in [8, Prop. 3.14] are the natural isomorphisms ${}_{\mathbb{Z}_p}\mathfrak{o} = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathfrak{o} \cong \prod_{\mathfrak{p}|p} \mathfrak{o}_{\mathfrak{p}}$ and ${}_{\mathbb{Q}_p}k = \mathbb{Q}_p \otimes_{\mathbb{Q}} k \cong \prod_{\mathfrak{p}|p} k_{\mathfrak{p}}$; we summarise the technical steps and implications in our setting. We write $\widetilde{\mathbf{H}}$ for the reductive part of the 1-component $\widetilde{\mathbf{G}}^{\circ}$. As described in Section 3.2, the local zeta function associated to the $\mathbb{Z}_p$-Lie lattice $\widetilde{L}_p = {}_{\mathbb{Z}_p}\widetilde{L}$ can be expressed as a p-adic integral

$$\zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s) = \int_{\widetilde{H}_p^+} |\det h|_p^s \, \vartheta_0(h)\, \vartheta_1(h) \, \mathrm{d}\mu_{\widetilde{H}_p}(h), \tag{7.8}$$

where $\widetilde{H}_p = \widetilde{\mathbf{H}}(\mathbb{Q}_p)$, $\widetilde{H}_p^+ = \widetilde{H}_p \cap \mathsf{M}_{nd}(\mathbb{Z}_p)$ and $\vartheta_0, \vartheta_1 \colon \widetilde{H}_p \to \mathbb{R}_{\geq 0}$ are suitable volume functions, modulo a small technical issue to be taken care of: while $\mathbf{G} = \mathbf{Aut}(L)$ is connected (as we proved), the group $\widetilde{\mathbf{G}}$ is typically not connected. But in the presence of (7.1) or the somewhat weaker condition (7.2), which we established in Section 7.1, the finite group scheme $\mathbf{F} \cong \mathbf{Aut}(\mathfrak{o}|\mathbb{Z}) \cong \mathbf{Aut}(k|\mathbb{Q})$, which potentially renders the group $\widetilde{\mathbf{G}}$ non-connected, has the feature that $\mathbf{F}(\mathbb{Z}_p) = \mathbf{F}(\mathbb{Q}_p)$ and can thus be safely ignored, by using the same argument as in the proof of [14, Prop. 2.1]. Moreover, the group $\widetilde{G}_p = \widetilde{\mathbf{G}}(\mathbb{Q}_p)$ almost, but not quite decomposes as a direct product indexed by the primes $\mathfrak{p} \mid p$. In the reductive part $\widetilde{H}_p$ the troublesome central automorphisms disappear and we have

$$\widetilde{H}_p \cong \prod_{\mathfrak{p}|p} H_{\mathfrak{p}} \quad \text{with } H_{\mathfrak{p}} = \mathbf{H}(k_{\mathfrak{p}}) \cong \widetilde{\mathbf{H}}_{\mathfrak{p}}(\mathbb{Q}_p) \text{ for each } \mathfrak{p} \mid p,$$

where $\mathbf{H}$ is the reductive part of the original group $\mathbf{G}$ and, setting $d_{\mathfrak{p}} = \dim_{\mathbb{Z}_p}(\mathfrak{o}_{\mathfrak{p}}) = [k_{\mathfrak{p}} : \mathbb{Q}_p]$, we denote by $\widetilde{\mathbf{H}}_{\mathfrak{p}}$ the reductive part of the algebraic automorphism group $\widetilde{\mathbf{G}}_{\mathfrak{p}} = \mathbf{Aut}(\widetilde{L}_{\mathfrak{p}}) \leq \mathsf{GL}_{nd_{\mathfrak{p}}}$ of the $\mathbb{Z}_p$-Lie lattice $\widetilde{L}_{\mathfrak{p}} = {}_{\mathbb{Z}_p,\mathfrak{o}_{\mathfrak{p}}}L$, which we analysed in Section 7.1. The next step is to transform the integral in (7.8) over $\widetilde{H}_p^+$ into a product of integrals over $H_{\mathfrak{p}}^+ = H_{\mathfrak{p}} \cap \mathsf{M}_n(\mathfrak{o}_{\mathfrak{p}})$ for $\mathfrak{p} \mid p$; this essentially uses the natural isomorphism between locally compact groups $(\mathsf{Res}_{\mathfrak{o}_{\mathfrak{p}} \mid \mathbb{Z}_p}(\mathbf{H}))(\mathbb{Q}_p) \cong \mathbf{H}(k_{\mathfrak{p}})$, but one also needs to pay attention to the accommodation of central automorphisms.

Modulo this small wrinkle, it is not difficult to carry out the analysis in Section 3.2 for the local field $k_\mathfrak{p}$ in place of $\mathbb{Q}_p$ to obtain

$$\zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s) = \prod_{\mathfrak{p}|p} \int_{H^+_\mathfrak{p}} |\det h|^s_\mathfrak{p}\, \vartheta_0(h)\, \vartheta_1(h)^d \,\mathrm{d}\mu_{H_\mathfrak{p}}(h), \tag{7.9}$$

where, for each $\mathfrak{p} \mid p$, the volume functions ϑ_0, ϑ_1 are defined in analogy to Section 3.2 (we refrain from adding the decoration '$\mathfrak{p}$') and $\mu_{H_\mathfrak{p}}$ denotes the right Haar measure on $H_\mathfrak{p}$ with the normalisation $\mu_{H_\mathfrak{p}}(H_\mathfrak{p}(\mathfrak{o}_\mathfrak{p})) = 1$; compare with the discussion in [8, §3], in particular with [8, Prop. 3.14]. It is worth pointing out that on the right-hand side of (7.9) the exponent of $\vartheta_1(h)$ is $d = [k : \mathbb{Q}]$ and not the corresponding local parameter $d_\mathfrak{p}$; this feature results from the treatment of central automorphisms and justifies that we consider the finite product of integrals as one 'package'.

It remains to carry out the explicit calculation of the integrals in (7.9) arising from the concrete cases $L = L_{t^2}$ and $L = L_{t^3}$. Consider first $L = L_{t^2}$. The calculation of the integral in Section 4 carries over with little change. The only material difference is that $\vartheta_1(h)$ in the integrand is replaced by $\vartheta_1(h)^d$. The intermediate integral (4.2) now takes the form

$$\int_{\substack{(A,\nu)\in\dot{H}_\mathfrak{p} \text{ with} \\ v_\mathfrak{p}(A)\geq 0,\, v_\mathfrak{p}(A)+v_\mathfrak{p}(\nu)\geq 0}} |\det A|^{4s-8d-2}_\mathfrak{p}\, |\nu|^{5s-12d}_\mathfrak{p} \,\mathrm{d}\mu_\mathfrak{p}(A,\nu)$$

where $\dot{H}_\mathfrak{p} = \dot{\mathbf{H}}(k_\mathfrak{p}) = \mathsf{GL}_2(k_\mathfrak{p}) \times \mathsf{GL}_1(k_\mathfrak{p})$ and the valuation map $v_\mathfrak{p}$ on $k_\mathfrak{p}$ and on $\mathsf{M}_2(k_\mathfrak{p})$ replaces the p-adic valuation v_p used previously. Due to the dependence on d, we then obtain

$$\vartheta_0(\xi_\mathbf{e}(\pi)^\varrho)\, \vartheta_1(\xi_\mathbf{e}(\pi)^\varrho)^d = q_\mathfrak{p}^{(8d+2)e_1+12de_2+(4d+4)e_3},$$

where π now denotes a uniformising element for $k_\mathfrak{p}$, that is $v_\mathfrak{p}(\pi) = 1$, and where $q_\mathfrak{p}$ denotes the residue field size of $k_\mathfrak{p}$, that is $q_\mathfrak{p} = |\mathfrak{o}/\mathfrak{p}| = |\pi|^{-1}_\mathfrak{p}$. We obtain the formula

$$\zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s) = \prod_{\mathfrak{p}|p} \frac{1 + q_\mathfrak{p}^{8d+2-4s}}{(1 - q_\mathfrak{p}^{4d+4-3s})(1 - q_\mathfrak{p}^{8d+3-4s})(1 - q_\mathfrak{p}^{12d-5s})}.$$

This is the local pro-isomorphic zeta function, at the prime p, of the class-two nilpotent group $\widetilde{\Gamma} = \widetilde{\Gamma}_{t^2}$ associated to $\widetilde{L} = \widetilde{L}_{t^2}$, as described in Section 3.1. It is straightforward to deduce Theorem 1.6 and the assertions in Remark 1.7.

Finally we consider $L = L_{t^3}$. The calculation of the integral in Section 5 carries over with little change. Indeed, the treatment there was already performed so that it applies equally well to the more general situation. Again, the only material difference is that $\vartheta_1(h)$ in the integrand is replaced by $\vartheta_1(h)^d$. The intermediate integral (5.5) now takes the form

$$\int_{\substack{(A,\nu)\in\dot{H} \text{ with} \\ v_{\mathfrak{p}}(A)\geq 0,\, v_{\mathfrak{p}}(A)+v_{\mathfrak{p}}(\nu)\geq 0}} |\det A|_{\mathfrak{p}}^{5s-12d}\, |\nu|_{\mathfrak{p}}^{-s+6d} \vartheta_0((A,\nu)^{\varrho})\, \mathrm{d}\mu_{\mathfrak{p}}(A,\nu)$$

where $\dot{H}_{\mathfrak{p}} = \dot{\mathbf{H}}(k_{\mathfrak{p}}) = \mathsf{GL}_2(k_{\mathfrak{p}}) \times \mathsf{GL}_1(k_{\mathfrak{p}})$ and the valuation map $v_{\mathfrak{p}}$ on $k_{\mathfrak{p}}$ and on $\mathsf{M}_2(k_{\mathfrak{p}})$ replaces the p-adic valuation v_p used previously. Due to the dependence on d, we have $\vartheta_1(\xi_{\mathbf{e}}(\pi)^{\varrho})^d = q_{\mathfrak{p}}^{12de_1+5de_2+18de_3}$, where π is a uniformising element for $k_{\mathfrak{p}}$ and where $q_{\mathfrak{p}}$ denotes the residue field size of $k_{\mathfrak{p}}$. Equation (5.22) becomes

$$\mathcal{Z}_{\mathfrak{p}}(s) = \mathcal{Z}_{\dot{\mathbf{H}},\varrho,\vartheta,\mathfrak{p}}(s) = \sum_{w\in W} q_{\mathfrak{p}}^{-\operatorname{len}(w)} \sum_{\substack{\mathbf{e}\in\mathcal{C} \text{ with} \\ e_1>0 \text{ if } w\neq 1}} X_1^{e_1} X_2^{e_2} X_3^{e_3}\, \tilde{\vartheta}(\xi_{\mathbf{e}}(\pi)),$$

where the numerical data is now

$$X_1 = q_{\mathfrak{p}}^{12d+2-5s}, \quad X_2 = q_{\mathfrak{p}}^{6d-4-s}, \quad X_3 = q_{\mathfrak{p}}^{18d+8-9s}.$$

The remaining calculations go through unchanged: the analogue of equation (5.25) yields the formula

$$\zeta_{\widetilde{L}_p}^{\mathrm{iso}}(s) = \prod_{\mathfrak{p}|p} \frac{(1-q_{\mathfrak{p}}^{24d+5-10s})V_{\mathfrak{p}}(s)}{(1-q_{\mathfrak{p}}^{12d+3-5s})^2(1-q_{\mathfrak{p}}^{18d+11-9s})(1-q_{\mathfrak{p}}^{30d-11s})(1-q_{\mathfrak{p}}^{54d+7-21s})}, \tag{7.10}$$

where

$$\begin{aligned} V_{\mathfrak{p}}(s) =& \frac{-q_{\mathfrak{p}}^{90d+14-36s} - q_{\mathfrak{p}}^{78d+12-31s} - q_{\mathfrak{p}}^{66d+9-26s} - q_{\mathfrak{p}}^{54d+5-21s}}{1+q_{\mathfrak{p}}^{12d+3-5s}} \\ &+ \frac{q_{\mathfrak{p}}^{36d+9-15s} + q_{\mathfrak{p}}^{24d+5-10s} + q_{\mathfrak{p}}^{12d+2-5s} + 1}{1+q_{\mathfrak{p}}^{12d+3-5s}} \\ =& 1 + q_{\mathfrak{p}}^{12d+2-5s} - q_{\mathfrak{p}}^{12d+3-5s} + q_{\mathfrak{p}}^{24d+6-10s} - q_{\mathfrak{p}}^{54d+5-21s} \\ &+ q_{\mathfrak{p}}^{66d+8-26s} - q_{\mathfrak{p}}^{66d+9-26s} - q_{\mathfrak{p}}^{78d+11-31s}. \end{aligned} \tag{7.11}$$

This is the local pro-isomorphic zeta function, at the prime p, of the class-two nilpotent group $\widetilde{\Gamma} = \widetilde{\Gamma}_{t^3}$ associated to $\widetilde{L} = \widetilde{L}_{t^3}$, as described in Section 3.1. We formulate, in analogy to Theorem 1.6, a partial generalisation of Corollary 1.4.

THEOREM 7.5: *Let k be a number field of absolute degree $d = [k : \mathbb{Q}]$, with ring of integers $\mathfrak{o}$. Let $\widetilde{\Gamma} = \widetilde{\Gamma}_{t^3,k}$ be the class-two nilpotent group of Hirsch length $8d$ and with rank-$2d$ centre, corresponding to the class-two nilpotent $\mathbb{Z}$-Lie lattice $\widetilde{L} = \widetilde{L}_{t^3,k}$ which results from the Lie lattice $L = L_{t^3}$ via 'base extension' as defined above.*

Then the pro-isomorphic zeta function of the group $\widetilde{\Gamma}$ is

$$\zeta^{\wedge}_{\widetilde{\Gamma}}(s) = \frac{\zeta_k(5s-(12d+3))^2\zeta_k(9s-(18d+11))\zeta_k(11s-30d)\zeta_k(21s-(54d+7))}{\zeta_k(10s-(24d+5))}\omega(s),$$

where $\zeta_k(s)$ denotes the Dedekind zeta function of k and

$$\omega(s) = \prod_{\mathfrak{p}} V_{\mathfrak{p}}(s) \tag{7.12}$$

with the product running over all non-archimedean primes $\mathfrak{p}$ of k and $V_{\mathfrak{p}}(s)$ defined as in (7.11).

Remark 7.6: For $k = \mathbb{Q}$, i.e., $d = 1$, the description is in agreement with Corollary 1.4; compare (6.1). Similar to the special situation covered in Section 6, it is routine to check that the infinite product in (7.12) converges absolutely and yields a holomorphic function on the half-plane consisting of all $s \in \mathbb{C}$ with

$$\begin{aligned}\mathrm{Re}(s) &> \max\left\{\frac{12d+4}{5}, \frac{24d+7}{10}, \frac{54d+6}{21}, \frac{66d+10}{26}, \frac{78d+12}{31}\right\} \\ &= \begin{cases} \frac{12d+4}{5} & \text{if } d \in \{1,2\}, \\ \frac{18d+2}{7} & \text{if } d \geq 3. \end{cases}\end{aligned}$$

Consequently, for number fields k of absolute degree $d \geq 3$, the pro-isomorphic zeta function of $\widetilde{\Gamma} = \widetilde{\Gamma}_{t^3,k}$ has abscissa of convergence $(30d+1)/11$ and can be meromorphically continued at least to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > (18d+2)/7\}$ with a simple pole at $s = (30d+1)/11$. For quadratic fields k, i.e., $d = 2$, there is an extra twist, but a routine analysis shows that the pro-isomorphic zeta function has abscissa of convergence $28/5$ and can be meromorphically continued at least to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 11/2\}$ with a simple pole at $s = 28/5$. Similar to Remark 1.5, the asymptotic growth of pro-isomorphic subgroups in $\widetilde{\Gamma}$ can be described by means of a suitable Tauberian theorem. Via the Euler product, the formula for $\zeta^{\wedge}_{\widetilde{\Gamma}}(s)$ incorporates the description (7.10) of the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\widetilde{\Gamma},p}(s) = \zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s)$ for all primes p and thus also yields a generalisation of

Theorem 1.3. Indeed, for $d = 2$ the zeta function $\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)$ has abscissa of convergence $115/21$ and for $d \geq 3$ it has abscissa of convergence $30d/11$. Whenever p is unramified in k, the local zeta function satisfies the functional equation

$$\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)|_{p\to p^{-1}} = \pm p^{24d^2+8d-10ds}\,\zeta^{\wedge}_{\widetilde{\Gamma},p}(s).$$

7.3. The local pro-isomorphic zeta functions of groups $\widetilde{\Gamma}$ associated to L_t. The Lie lattice $L = L_t$ is decomposable and does not quite fit into the same drawer as the lattices L_{t^m}, $m \geq 2$. For completeness we indicate how the approach in [8] can be adapted in this and similar situtations to obtain the local pro-isomorphic zeta functions of class-two nilpotent groups $\widetilde{\Gamma}$ associated to Lie lattices $\widetilde{L}$ obtained from L via 'base extension'.

We start our discussion more generally. Let L be any class-two nilpotent $\mathbb{Z}$-Lie lattice, and throughout let p denote a rational prime. Then $[L, L] \subseteq \mathrm{Z}(L)$ and we can decompose L as a direct sum $L = L^\circ \oplus M$ of Lie sublattices, where L° satisfies $[L^\circ, L^\circ] = \mathrm{Z}(L^\circ) = [L, L]$ and $M \subseteq \mathrm{Z}(L)$ is abelian. Typically there are many choices for L° and M, but both are uniquely determined up to isomorphism. We set $l = \dim_{\mathbb{Z}}(L^\circ/[L^\circ, L^\circ])$, $m = \dim_{\mathbb{Z}}(M)$ and $n = \dim_{\mathbb{Z}}([L^\circ, L^\circ])$. The algebraic automorphism group $\mathbf{Aut}(L^\circ)$ can be realised as a subgroup scheme $\mathbf{G} \leq \mathsf{GL}_{l+n}$ via a $\mathbb{Z}$-basis $\mathcal{S}^\circ = (x_1, \ldots, x_l, z_1, \ldots, z_n)$ for L° consisting of a basis $x_1, \ldots, x_l$ for a complement of $[L^\circ, L^\circ]$ in L° and a basis $z_1, \ldots, z_n$ for $[L^\circ, L^\circ]$. Similarly we view $\mathbf{Aut}(L)$ as a subgroup scheme of GL_{l+m+n} via the extended $\mathbb{Z}$-basis

$$\mathcal{S} = (x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n),$$

where $y_1, \ldots, y_m$ form a basis for M.

There are polynomial conditions, which we will denote by (†), and a polynomial map f from GL_l to GL_n, which can be made explicit in terms of the structure constants of the Lie lattice L°, such that automorphisms of the $\mathbb{Q}_p$-Lie algebra ${}_{\mathbb{Q}_p}L^\circ = \mathbb{Q}_p \otimes_{\mathbb{Z}} L^\circ$, viewed as elements of the group $G_p = \mathbf{G}(\mathbb{Q}_p) \leq \mathsf{GL}_{l+n}(\mathbb{Q}_p)$, take the form

$$\left(\begin{array}{c|c} A & * \\ \hline & f(A) \end{array}\right), \quad \text{where } A \in \mathsf{GL}_l(\mathbb{Q}_p) \text{ satisfies (†)} \tag{7.13}$$

and $*$ is a placeholder for arbitrary entries.

Moreover, automorphisms of ${}_{\mathbb{Q}_p}L = \mathbb{Q}_p \otimes_{\mathbb{Z}} L$, viewed as elements of $\mathsf{GL}_{l+m+n}(\mathbb{Q}_p)$, take the form

$$\left(\begin{array}{c|c|c} A & * & * \\ \hline & B & * \\ \hline & & f(A) \end{array}\right), \quad \begin{array}{l} \text{where } A \in \mathsf{GL}_l(\mathbb{Q}_p) \text{ satisfies } (\dagger), \\ B \in \mathsf{GL}_m(\mathbb{Q}_p) \text{ has no particular restrictions} \\ \text{and } * \text{ is a placeholder for arbitrary entries.} \end{array} \tag{7.14}$$

We set ${}_pL^\circ = \mathbb{Z}_p \otimes_{\mathbb{Z}} L^\circ$ and ${}_pL = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$. As described in Section 3.2, the local zeta function $\zeta^{\wedge}_{L^\circ,p}(s) = \zeta^{\mathrm{iso}}_{L^\circ_p}(s)$ is under suitable assumptions given by an integral

$$\zeta^{\mathrm{iso}}_{L^\circ_p}(s) = \int_{H_p^+} |\det h|_p^s \, \vartheta_0(h)\, \vartheta_1(h) \, \mathrm{d}\mu_{H_p}(h) \tag{7.15}$$

over $H_p^+ = H_p \cap \mathsf{M}_{l+n}(\mathbb{Z}_p)$, where H_p denotes the reductive part of G_p and $\vartheta_0, \vartheta_1 \colon H_p \to \mathbb{R}_{\geq 0}$ are suitable volume functions. The descriptions in (7.13) and (7.14) provide a close link between the groups of $\mathbb{Q}_p$-points of the algebraic automorphism groups of L° and L; for instance, the reductive parts are H_p and $H_p \times \mathsf{GL}_m(\mathbb{Q}_p)$, up to isomorphism. Utilizing this connection, we obtain for the local zeta function $\zeta^{\wedge}_{L,p}(s) = \zeta^{\mathrm{iso}}_{L_p}(s)$ the integral formula

$$\begin{aligned} \zeta^{\mathrm{iso}}_{L_p}(s) = &\int_{H_p^+} |\det h|_p^s \, \vartheta_0(h)\, \vartheta_1(h)^{(l+m)/l} \, \mathrm{d}\mu_{H_p}(h) \\ &\times \int_{\mathsf{GL}_m(\mathbb{Q}_p)^+} |\det g|_p^{s-l} \, \mathrm{d}\mu_{\mathsf{GL}_m(\mathbb{Q}_p)}(g), \end{aligned} \tag{7.16}$$

in analogy to (7.9); the first factor is a mild modification of the integral in (7.15) and accomodates for the extra middle block in the third column of (7.14), the second factor accommodates for the extra blocks in the middle column in (7.14). The second factor in (7.16) is well-known and easy to compute; one gets

$$\int_{\mathsf{GL}_m(\mathbb{Q}_p)^+} |\det g|_p^{s-l} \, \mathrm{d}\mu_{\mathsf{GL}_m(\mathbb{Q}_p)}(g) = \prod_{j=1}^{m} (1 - p^{(l+j-1)-s})^{-1}.$$

Example 7.7: Let $L = L^\circ \oplus M$, where

$$L^\circ = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 \oplus \mathbb{Z}z_1$$

with $[x_1, x_2] = z_1$ denotes the Heisenberg Lie lattice and

$$M = \bigoplus_{i=1}^{m} \mathbb{Z}y_i \cong \mathbb{Z}^m$$

is abelian. Then $(l,n) = (2,1)$ and

$$H_p = \{\mathrm{diag}(A, \det A) \mid A \in \mathsf{GL}_2(\mathbb{Q}_p)\} \leq \mathsf{GL}_3(\mathbb{Q}_p),$$

furthermore $\vartheta_0(h) = 1$ and $\vartheta_1(h) = |\det A|_p^{-2}$ for $h = \mathrm{diag}(A, \det A) \in H_p$ in the above approach; consequently, we obtain

$$\begin{aligned}\zeta^{\mathrm{iso}}_{L_p}(s) &= \int_{\mathsf{GL}_2(\mathbb{Q}_p)^+} |\det A|_p^{2s-m-2} \,\mathrm{d}\mu_{\mathsf{GL}_2(\mathbb{Q}_p)}(h) \cdot \prod_{j=1}^{m} (1-p^{(j+1)-s})^{-1} \\ &= (1-p^{(m+2)-2s})^{-1}(1-p^{(m+3)-2s})^{-1} \prod_{j=1}^{m} (1-p^{(j+1)-s})^{-1},\end{aligned}$$

in agreement with the calculation in [3, §3.3.4]. The pro-isomorphic zeta function of the corresponding group $\Gamma \cong \mathrm{Heis}(\mathbb{Z}) \times C_\infty^m$ is a product of shifted Riemann zeta functions.

In a situation where L° satisfies rigidity conditions of the kind described in Section 7.1 the approach discussed in Section 7.2 can easily be adapted to yield a formula for the local zeta functions $\zeta^{\wedge}_{\widetilde{L},p}(s) = \zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s)$ of Lie lattices $\widetilde{L}$ obtained from L via 'base extension'. As before let k be a number field of absolute degree $d = [k : \mathbb{Q}]$, with ring of integers $\mathfrak{o}$. We apply the method and notation from Section 7.2 to L° in place of L. Let $\widetilde{L} = {}_{\mathbb{Z},\mathfrak{o}}L$ be the class-two nilpotent $\mathbb{Z}$-Lie lattice obtained by extending and restricting scalars. We observe that $\widetilde{L} = \widetilde{L^\circ} \oplus \widetilde{M}$ is a suitable decomposition of $\widetilde{L}$ in the sense given at the beginning of this section, with $\widetilde{L}^\circ = \widetilde{L^\circ}$ satisfying $[\widetilde{L}^\circ, \widetilde{L}^\circ] = [\widetilde{L}, \widetilde{L}]$ and $\widetilde{M} \subseteq \mathrm{Z}(\widetilde{L})$ abelian; furthermore, $\dim_{\mathbb{Z}}(\widetilde{L}^\circ) = (l+n)d$ and $\dim_{\mathbb{Z}}(\widetilde{M}) = md$. Combining the approaches taken in this and the previous section, we arrive at the integral formula

$$\begin{aligned}\zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s) =& \Big(\prod_{\mathfrak{p}\mid p} \int_{H_\mathfrak{p}^+} |\det h|_\mathfrak{p}^s \,\vartheta_0(h)\,\vartheta_1(h)^{(l+m)d/l} \,\mathrm{d}\mu_{H_\mathfrak{p}}(h) \Big) \\ &\times \int_{\mathsf{GL}_{md}(\mathbb{Q}_p)^+} |\det g|_p^{s-ld} \mathrm{d}\mu_{\mathsf{GL}_{md}(\mathbb{Q}_p)}(g) \\ =& \Big(\prod_{\mathfrak{p}\mid p} \int_{H_\mathfrak{p}^+} |\det h|_\mathfrak{p}^s \,\vartheta_0(h)\,\vartheta_1(h)^{(l+m)d/l} \,\mathrm{d}\mu_{H_\mathfrak{p}}(h) \Big) \prod_{j=1}^{md} (1-p^{(ld+j-1)-s})^{-1},\end{aligned}$$

where $H_\mathfrak{p}$, $H_\mathfrak{p}^+$, ϑ_0 and ϑ_1 (again without the decoration '$\mathfrak{p}$') are defined as in preparation for (7.9), but applied to L° instead of L.

Finally we apply the above considerations to $L = L_t = L^\circ \oplus M$, where

$$L^\circ = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 \oplus \mathbb{Z}z_1$$

with $[x_1, x_2] = z_1$ is the Heisenberg Lie lattice and

$$M = \mathbb{Z}y_1$$

is abelian of rank 1. In this case $(l, m, n) = (2, 1, 1)$ and our general formula specialises to

$$\begin{aligned}\zeta^{\mathrm{iso}}_{\widetilde{L}_p}(s) &= \left(\prod_{\mathfrak{p}|p} \int_{\mathsf{GL}_2(k_\mathfrak{p})^+} |\det A|_\mathfrak{p}^{2s-3d} \, \mathrm{d}\mu_{\mathsf{GL}_2(k_\mathfrak{p})}(h)\right) \cdot \prod_{j=1}^{d} (1 - p^{(2d+j-1)-s})^{-1} \\ &= \left(\prod_{\mathfrak{p}|p} (1 - q_\mathfrak{p}^{3d-2s})^{-1} (1 - q_\mathfrak{p}^{(3d+1)-2s})^{-1}\right) \cdot \prod_{j=1}^{d} (1 - p^{(2d+j-1)-s})^{-1},\end{aligned}$$

where $q_\mathfrak{p}$ denotes the residue field size of $k_\mathfrak{p}$. The pro-isomorphic zeta function of the corresponding class-two nilpotent group $\widetilde{\Gamma}$ is a product of two shifts of the Dedekind zeta function of k and d shifted Riemann zeta functions.

THEOREM 7.8: *Let k be a number field of absolute degree $d = [k : \mathbb{Q}]$, with ring of integers $\mathfrak{o}$. Let $\widetilde{\Gamma} = \widetilde{\Gamma}_{t,k}$ be the class-two nilpotent group of Hirsch length $4d$ and with rank-$2d$ centre, corresponding to the class-two nilpotent $\mathbb{Z}$-Lie lattice $\widetilde{L} = \widetilde{L}_{t,k}$ which results from the Lie lattice $L = L_t$ via 'base extension' as defined above.*

Then the pro-isomorphic zeta function of the group $\widetilde{\Gamma}$ is

$$\zeta^{\wedge}_{\widetilde{\Gamma}}(s) = \zeta_k(2s - 3d)\, \zeta_k(2s - (3d + 1)) \cdot \prod_{j=1}^{d} \zeta_{\mathbb{Q}}(s - (2d + j - 1)),$$

where $\zeta_k(s)$ denotes the Dedekind zeta function of k and $\zeta_{\mathbb{Q}}(s)$ denotes the Riemann zeta function; in particular, it admits meromorphic continuation to the entire complex plane.

Remark 7.9: The abscissa of convergence of $\zeta^{\wedge}_{\widetilde{\Gamma}}(s)$ is $3d$, with a single pole at $s = 3d$. The asymptotic growth of pro-isomorphic subgroups in $\widetilde{\Gamma}$ can be described by means of a suitable Tauberian theorem. The local zeta function $\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)$ has abscissa of convergence $3d - 1$ and, if p is unramified in k, it satisfies the functional equation

$$\zeta^{\wedge}_{\widetilde{\Gamma},p}(s)|_{p \to p^{-1}} = \pm p^{8d^2 + \frac{1}{2}d(d+1) - 5ds} \, \zeta^{\wedge}_{\widetilde{\Gamma},p}(s).$$

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